ON FIXED-POINT THEOREMS IN SYNTHETIC COMPUTABILITY

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ABSTRACT. Lawvere's fixed point theorem captures the essence of diagonalization arguments. Cantor's theorem, Gödel's incompleteness theorem, and Tarski's undefinability of truth are all instances of the contra-positive form of the theorem. It is harder to apply the theorem directly because non-trivial examples are not easily found, in fact, none exist if excluded middle holds.

We study Lawvere's fixed-point theorem in the effective topos. Rather than operating directly with the topos, we work in synthetic computability, which is higher-order intuitionistic logic augmented with the Axiom of Countable Choice, Markov's principle, and the Enumeration axiom, which states that there are countably many countable subsets of $\mathbb{N}$. These extra-logical principles are valid in the effective topos, as well as in any realizability topos built over Turing machines with an oracle, and suffice for an abstract axiomatic development of a computability theory.

We show that every countably generated ω-chain complete pointed partial order (ωcppo) is countable, and that countably generated ωcppos are closed under countable products. Therefore, Lawvere's fixed-point theorem applies and we obtain fixed points of all endomaps on countably generated ωcppos. Similarly, the Knaster–Tarski theorem guarantees existence of least fixed points of continuous endomaps. To get the best of both theorems, we prove a synthetic version of the Myhill–Shepherdson theorem: every map from an ωcpo to a domain (an ωcppo which is generated by a countable set of compact elements) is continuous. The proof relies on a new fixed-point theorem, the synthetic Recursion Theorem. It subsumes the classic Kleene-Rogers Recursion Theorem, and takes the form of Lawvere's fixed point theorem for multi-valued endomaps.

1. INTRODUCTION

A fixed point theorem of Lawvere's [12, Theorem 1.1] is the quintessential diagonal argument. The following version uses a stronger notion of surjectivity than the original theorem, but its statement and proof may be interpreted in the internal language of a topos.

Theorem 1.1 (Lawvere). *If there is a surjection $e : A \to B^A$ then every map $f : B \to B$ has a fixed point.*

Proof. There is $a \in A$ such that $e(a) = \lambda x{:}A\,.\,f(e(x)(x))$, thus $e(a)(a) = f(e(a)(a))$. □

Among the consequences of Lawvere's original theorem are Cantor's theorem [12, Corollary 1.2], Gödel's incompleteness theorem [12, Theorem 3.3], and Tarski's undefinability of truth [12, Theorem 3.2]. These all take the contrapositive form: because some object has an endomap without fixed points, some surjection does not exist. For instance, the contrapositive form of Theorem 1.1 implies that there is no surjection from A to its power Ω^A, because negation has no fixed points as an endomap on the object of truth values Ω.

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How about direct applications of the theorem? None are given by Lawvere [12], and there are none in the presence of excluded middle, where a surjection $A \to B^A$ is possible only if B is a singleton. We shall heed Lawvere's advice and look for them in computability theory.

Rather than crafting Turing machines and Gödel codes, we prefer to work in Hyland's effective topos [6]. In fact, we shall avoid chasing diagrams, too, and instead use exclusively the internal language of the topos. To be precise, our settings is higher-order intuitionistic logic with a natural numbers object [11], augmented with just three extra-logical principles: the Axiom of Countable Choice, the Enumeration Axiom, and Markov's Principle; see §2 for their formulation and explanation. The principles are valid in the effective topos, as well as in any realizability topos arising from a partial combinatory algebra of (codes of) Turing machines with an oracle. In other words, all results relativize with respect to an arbitrary oracle. I call this setup *synthetic computability* because it is grounded in a 'synthetic' mathematical universe with computability theory built in, but it approaches the subject in an axiomatic and abstract way that eschews talking about computation explicitly. It is similar in techniques and ideas to synthetic domain theory [19, 7, 16, 21] and synthetic topology [5, 13, 22, 1], with an emphasis on topics that pertain to computability theory.

We shall seek examples of Lawvere's theorem in domain theory. For this purpose we review in §3 the basic definitions and facts about ω-complete partial orders (ωcpos) and their pointed versions, ωcppos, and make sure that everything works in intuitionistic logic with the Axiom of Countable Choice (but we do not use Markov's principle or the Enumeration Axiom). Of special interest are the countably generated ωcppos and domains, both of which are defined in §3.

In §4 we fulfill our initial task by proving with the help of the Enumeration Axiom that all countably generated ωcppos satisfy Lawvere's fixed-point theorem. In fact, they satisfy two two fixed-point theorems:

(1) Every endomap has a fixed point.
(2) Every continuous endomap has a least fixed point.

The first one is Lawvere's theorem, and the second one the Knaster–Tarski theorem. In order to reconcile these into a single theorem we develop more synthetic computability in §5. We formulate and prove a new version of Lawvere's fixed point theorem (Theorem 5.2), which states the existence of fixed points of multi-valued maps. We call it the Recursion Theorem because it implies the classic Kleene-Rogers Recursion Theorem, and allows us to construct various recursive objects.

The Recursion Theorem applies to all countably generated ωcppos. We use it to prove a continuity principle (Theorem 6.4) stating that all maps from ωcpos to domains are continuous. The principle subsumes other continuity principles, such as the classic Myhill-Shepherdson theorem and Scott's principle from synthetic domain theory. It also implies that all endomaps on domains have least fixed points, and so at least for domains the reconciliation of fixed-point theorems is accomplished.

2. SYNTHETIC COMPUTABILITY

We shall work in higher-order intuitionistic logic with a natural numbers object [11], enriched with three extra-logical principles. The first one is the Axiom of Countable Choice:

Axiom 2.1. *A total relation whose domain is $\mathbb{N}$ contains the graph of a function.*

Written in intuitionistic higher-order logic, the axiom states that for every relation $\psi : \mathbb{N} \times A \to \Omega$,

$$(\forall n \in \mathbb{N} \,.\, \exists x \in A \,.\, \psi(x, a)) \Rightarrow \exists f \in A^{\mathbb{N}} \,.\, \forall n \in \mathbb{N} \,.\, \psi(n, f(n)).$$

The second axiom is Markov's principle [14]:

Axiom 2.2. *If a binary sequence is not constantly* 0 *then it contains a* 1.

Written as a formula, Markov's principles states

$$\forall \alpha \in 2^{\mathbb{N}} \,.\, \neg(\forall n \,.\, \alpha_n = \top) \Rightarrow \exists n \,.\, \alpha_n = \top.$$

where $2 = \{p \in \Omega \mid p \vee \neg p\}$ is the set of decidable truth values. In Proposition 3.4 we shall see another formulation of Markov's principle.

The Axiom of Countable Choice and Markov's principle are both valid in the effective topos, as was noted already in [6]. The third tenet is the *Enumeration Axiom*:

Axiom 2.3. *There are countably many countable subsets of the natural numbers.*

Let us be more precise. A set A is *countable*, or *enumerable*, if there is a surjection $e : \mathbb{N} \to 1 + A$, where summing the codomain with the singleton $1 = \{\star\}$ allows an enumeration to 'skip' by outputting $\star$. Thus the empty set is enumerated by a sequence that always skips. When A is inhabited there is a surjection $1 + A \to A$ that provides an enumeration $\mathbb{N} \to A$ without skipping. The countable subsets of a set A are the restrictions of images of maps $\mathbb{N} \to 1 + A$ to A, and they again form a set

$$\mathcal{E}(A) = \{S \in \Omega^A \mid \exists e \in (1 + A)^{\mathbb{N}} \,.\, \forall x \in A \,.\, x \in S \Leftrightarrow \exists n \,.\, e(n) = x\}.$$

If we let $\mathcal{E}$ abbreviate $\mathcal{E}(\mathbb{N})$ then the Enumeration Axiom states that there is a surjection

$$\mathsf{W} : \mathbb{N} \to \mathcal{E}.$$

We are using standard notation for computability theory because it suggests how the effective topos validates the Enumeration axiom: $\mathcal{E}$ is just the object of computably enumerable sets. In Ershov's theory of numbered sets [4] it is the numbered set $(\mathcal{E}, \mathsf{W})$, where $\mathcal{E}$ is the set of computably enumerable sets and $\mathsf{W} : \mathbb{N} \to \mathcal{E}$ a standard numbering of $\mathcal{E}$. The Enumeration axiom is valid simply because W is total, and so the realizer for surjectivity of W is just the the identity map. After we have developed some theory, in §4.1 we shall compare the Enumeration axiom to other axiomatic formulations of computability theory.

We use the Axiom of Countable Choice frequently, Markov's principle only in Proposition 6.1, and the Enumeration axiom only in Theorem 4.2.

Higher-order intuitionistic logic can be put to work immediately, even without the extra axioms. The contra-positive form of Lawvere's theorem tells us that there are no surjections

$$\mathbb{N} \to \mathbb{N}^{\mathbb{N}} \qquad \text{and} \qquad \mathbb{N} \to 2^{\mathbb{N}}$$

because the natural numbers and the decidable truth values have endomaps without fixed points, namely the successor and the negation, respectively. The realizability interpretations of these statements are the familiar facts that there are no computable enumerations of total computable functions and of computable subsets of $\mathbb{N}$, respectively.

A slightly more interesting observation is the following proposition, which will serve to prove a synthetic variant of Rice's theorem, see Corollary 4.4. Say that a set has the *fixed-point property* if every endomap on it has a fixed point.

Proposition 2.4. *If A has the fixed point property then every map $A \to 2$ is constant.*

Proof. Given $f : A \to 2$ and any $x, y \in A$ we show that $f(x) = f(y)$. Define $g : A \to A$ by

$$g(z) = \begin{cases} x & \text{if } f(z) = f(y), \\ y & \text{otherwise.} \end{cases}$$

There is $z \in A$ such that $z = f(z)$. If $f(z) = f(y)$ then $z = g(z) = x$ and $f(x) = f(z) = f(y)$. If $f(z) \neq f(y)$ then $z = g(z) = y$ and so $f(z) = f(y)$, a contradiction, hence again $f(x) = f(y)$. □

3. Chain-complete pointed partial orders

We shall look for sets that satisfy the precondition of Lawvere's theorem in domain theory. We first review the relevant concepts, and make sure that they work intuitionistically.

A *partially ordered set*, or *poset*, $(P, \leq)$ is a set P with a reflexive, transitive and asymmetric relation $\leq$. A *chain* in $(P, \leq)$ is a a monotone sequence $c : \mathbb{N} \to P$: for all $i \in \mathbb{N}$, $c_i \leq c_{i+1}$. A *chain-complete poset (ωcpo)* is a poset $(P, \leq)$ in which every chain $c : \mathbb{N} \to P$ has a supremum $\bigvee_n c_n$. If an ωcpo has a least element $\bot$, called the *bottom*, then it is a *pointed ωcpo (ωcppo)*. A map between ωcpos is *continuous* when it is monotone and it preserves suprema of chains.

A *countable base*, or just a *base*, for an ωcppo $(P, \leq)$ is a countable subset $B \subseteq P$, whose elements are called *basic*, such that:

(1) every element in P is the supremum of a chain of basic elements, and
(2) the induced order on B is decidable.

A *countably generated* ωcppo is one that has a base. Note that our terminology differs from the established one, as a base often involves the way below relation. The bottom $\bot$ is always basic because it is the supremum of a chain of basic elements, but those must all be $\bot$.

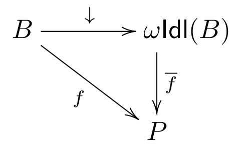
We must thread still slightly deeper into domain theory. In an ωcppo $(P, \leq)$ an element $x \in P$ is *compact* if, for every countable chain $c : \mathbb{N} \to P$, $x \leq \bigvee_n c_n$ implies $x \leq c_n$ for some $n \in \mathbb{N}$. A *domain* is a countably generated ωcppo whose basic elements are compact. If basic elements are compact then all compact elements are basic, for each compact element is the supremum of a chain of basic elements, and therefore equal to one of them by compactness.

Domains are generally constructed as completions. Let $(B, \leq)$ be a countable poset with a least element $\bot$ and a decidable order. An *ideal* in B is a subset $I \subseteq B$ which is

(1) *inhabited*: $\bot \in I$,
(2) *downward closed*: if $x \leq y$ and $y \in I$ then $x \in I$, and
(3) *directed*: for all $x, y \in I$ there is $z \in I$ such that $x \leq z$ and $y \leq z$.

The poset $\omega\mathsf{Idl}(B)$ of *countable* ideals in B ordered by inclusion $\subseteq$, is a domain. The least element is the trivial ideal $\{\bot\}$, and the supremum of a countable chain of countable ideals is again a countable ideal, thanks to the Axiom of Countable Choice. For each $x \in B$ the principal ideal ${\downarrow}x = \{y \in B \mid y \leq x\}$ is countable because the order on B is decidable, and it is compact in $\omega\mathsf{Idl}(B)$. Thus principal ideals form a countable base of compact elements. The construction is a completion because it has the following universal property.

Proposition 3.1. *For any countable poset $(B, \leq_B)$ with a least element and decidable order, and a monotone map $f : B \to P$ into an ωcpo $(P, \leq_P)$, there exists a unique* *continuous extension* $\overline{f} : \omega\mathsf{Idl}(B) \to P$ *such that the following diagram commutes:*



Proof. Let us first show that every countable ideal $I \subseteq B$ contains a countable chain which is cofinal in I. Starting from an enumeration $e : \mathbb{N} \to I$, by the Axiom of Countable Choice there is a map $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which chooses for any $m, n \in \mathbb{N}$ an element $s(m, n) \in \mathbb{N}$ such that $e_m \leq_B e_{s(m,n)}$ and $e_n \leq_B e_{s(m,n)}$. Let $r : \mathbb{N} \to \mathbb{N}$ be defined by $r(0) = 0$ and $r(n + 1) = s(r(n), n + 1)$. Then $c = e \circ r$ is the cofinal chain we are looking for. Therefore, I is the supremum of the chain of principal ideals ${\downarrow}c_n$, and so it must be the case that

$$\overline{f}(I) = \overline{f}(\textstyle\bigvee_n {\downarrow}c_n) = \bigvee_n \overline{f}({\downarrow}c_n) = \bigvee_n f(c_n).$$

The right-hand side does not depend on the cofinal chain $c : \mathbb{N} \to I$, because any two cofinal chains in I dominate each other, and so do their images by f. Thus the equation may be taken as the definition of $\overline{f}$. Uniqueness now follows by a standard argument. □

We really do have to take only the countable ideals, because arbitrary ideals may not contain cofinal chains. For example, the ideal completion of the finite decidable subsets of $\mathbb{N}$ is the powerset $\Omega^{\mathbb{N}}$. If every subset of $\mathbb{N}$ were the union of a chain of finite sets, then every subset of $\mathbb{N}$ would be countable, $\Omega^{\mathbb{N}} = \mathcal{E}$, but this cannot be because Cantor's theorem states uncountability of $\Omega^{\mathbb{N}}$ and the Enumeration axiom countability of $\mathcal{E}$.

Countably generated ωcppos are closed for several construction, but of special interest to us is closure for countable products.

Theorem 3.2. *Countably generated* ω*cppos are closed under countable products, and so are domains.*

Proof. Let $(P_i, \leq_i)_{i\in\mathbb{N}}$ be a sequence of countably generated ωcppos and $Q = \prod_{i\in\mathbb{N}} P_i$ their product, ordered coordinate-wise. We need to exhibit a base for Q. By the Axiom of Countable Choice, for every $i \in \mathbb{N}$ there is an enumeration $b_i : \mathbb{N} \to P_i$ of a base for P_i. Let $\mathsf{List}(\mathbb{N})$ be the set of finite lists of numbers, let $|s|$ be the length of the list s, and s_i its i-th element, counting from zero. There is an enumeration $\ell : \mathbb{N} \to \mathsf{List}(\mathbb{N})$. Define $c : \mathbb{N} \to Q$ by

$$c_n(i) = \begin{cases} b_i(\ell(n)_i) & \text{if } i < |\ell(n)|, \\ \bot_i & \text{otherwise.} \end{cases}$$

One readily verifies that c is a base for Q.

To prove the statement for domains, one just has to additionally verify that c enumerates compact elements in Q if the b_i's do the same in P_i's. □

Below we give examples of domains, but counter-examples are equally instructive. The closed interval $[0, 1]$ with the usual ordering $\leq$ is *not* closed under suprema of chains, because such closure would imply the Limited Principle of Omniscience, which is false, see Corollary 4.3 and the subsequent paragraph. However, the related closed interval $[0, 1]_\ell$ of *lower* reals is a ωcppo. (Recall that the lower reals are constructed as the set of lower Dedekind cuts, whereas the construction of reals uses two-sided cuts.) The rational numbers between 0 and 1 form a countable base for $[0, 1]_\ell$, but $[0, 1]_\ell$ is not a domain, since its only compact element is 0.

3.1. **Lifting and partial maps.** Given a countable set A with decidable equality, take as the base the poset $A + \{\bot\}$ in which $x \leq y$ if, and only if, $x = \bot$ or $x = y$. Its completion $\omega\mathsf{Idl}(A + \{\bot\})$ is a domain known as the *lifting* $A_\bot$. It may also be described as the poset of countable subsets in A with at most one element,

$$A_\bot = \{S \in \mathcal{E}(A) \mid \forall x, y \in S \,.\, x = y\},$$

ordered by inclusion. A countable base for $A_\bot$ consists of those elements of $A_\bot$ which are either empty or inhabited.

Let $\tilde{A}$ be the set of subsets of A with at most one element,

$$\tilde{A} = \{S \in \Omega^A \mid \forall x, y \in S \,.\, x = y\}.$$

A *partial map* $f : A \rightharpoonup B$ is a map $f : A \to \tilde{B}$, where $f(x) = \emptyset$ signifies that f is undefined at x and $f(x) = \{y\}$ that it takes the value y. The *support* of f is

$$\|f\| = \{x \in A \mid \exists y \in B \,.\, f(x) = \{y\}\},$$

and its *graph* the set

$$\Gamma_f = \{(x, y) \in A \times B \mid f(x) = \{y\}\}.$$

Because $B_\bot \subseteq \tilde{B}$, every map $f : A \to B_\bot$ is a partial map.

Proposition 3.3. *The following are equivalent for a partial map $f : \mathbb{N} \rightharpoonup \mathbb{N}$:*

(1) *the support of f is countable,*
(2) *the graph of f is countable,*
(3) *the values of f are countable.*

Proof. Note that the last claim states that f may be seen as a map $\mathbb{N} \to \mathbb{N}_\bot$.

If $e : \mathbb{N} \to 1 + \|f\|$ enumerates the support then the graph is enumerated by

$$k \mapsto \begin{cases} (i, j) & \text{if } e(k) = i \text{ and } f(i) = \{j\}, \\ \star & \text{otherwise.} \end{cases}$$

If $e : \mathbb{N} \to 1 + \Gamma_f$ enumerates the graph then, for every $m \in \mathbb{N}$, the value $f(m)$ is enumerated by

$$k \mapsto \begin{cases} n & \text{if } e(k) = (m, n), \\ \star & \text{otherwise.} \end{cases}$$

Suppose $f(m)$ is countable for every $m \in \mathbb{N}$. By the Axiom of Countable Choice there is $e : \mathbb{N} \times \mathbb{N} \to 1 + \mathbb{N}$ such that $e(m, -)$ enumerates $f(m)$ for every $m \in \mathbb{N}$. Then the support of f is enumerated by

$$\langle m, k \rangle \mapsto \begin{cases} \star & \text{if } e(m, k) = \star, \\ m & \text{otherwise.} \end{cases}$$

where $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is any bijection. □

The proposition tells us that ${\mathbb{N}_\bot}^{\mathbb{N}}$ may be identified with partial maps whose graphs are countable. In the effective topos it is the object of partial computable maps, i.e., the numbered set of partial computable maps with a standard numbering. In § 4.1 we shall return to ${\mathbb{N}_\bot}^{\mathbb{N}}$ in relation to other axiomatizations of computability theory.

3.2. **The Rosolini dominance.** Of special interest is the *Rosolini dominance* [19], which can be described as the completion $\omega\mathsf{Idl}(2)$ of the Boolean lattice 2, or as the set of *semidecidable* truth values,

$$\Sigma = \{p \in \Omega \mid \exists q \in 2^{\mathbb{N}} \,.\, (p \Leftrightarrow \exists n \in \mathbb{N} \,.\, q_n = \top)\},$$

with the order induced by that of Ω. It has arbitrary countable suprema, not just those of chains, a fact whose proof relies on the Axiom of Countable Choice. The decidable truth values $2 = \{p \in \Omega \mid p \vee \neg p\}$ form a countable base for Σ.

Several logical principles may be expressed as relationships between sublattices of Ω. Let $\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p \Rightarrow p\}$ be the set of $\neg\neg$-stable truth values.

Proposition 3.4. *Excluded middle states that $2 = \Omega$, the Limited Principle of Omniscience that $2 = \Sigma$, and Markov's principle that $\Sigma \subseteq \Omega_{\neg\neg}$.*

Proof. Excluded middle states that, for all $p \in \Omega$, p or $\neg p$, which is equivalent to saying that $\Omega \subseteq 2$, while the reverse inclusion holds by definition of 2.

The Limited Principle of Omniscience [2] states that, for all $q : \mathbb{N} \to 2$, either $\exists n \in \mathbb{N} \,.\, q_n = \top$ or $\forall n \in \mathbb{N} \,.\, q_n = \bot$. The latter is equivalent to $\neg\exists n \in \mathbb{N} \,.\, q_n = \bot$, and so the principle is equivalent to $\forall p \in \Sigma \,.\, p \vee \neg p$, which may be expressed as $2 = \Sigma$.

Finally, Markov's principle is equivalent to the statement that, for all $q : \mathbb{N} \to 2$, if $\neg\neg\exists n \in \mathbb{N} \,.\, q_n = \top$ then $\exists n \in \mathbb{N} \,.\, q_n = \top$. This in turn is equivalent to $\forall p \in \Sigma \,.\, \neg\neg p \Rightarrow p$, which may be expressed as $\Sigma \subseteq \Omega_{\neg\neg}$. □

The Rosolini dominance classifies the countable subsets of $\mathbb{N}$ so that

$$\mathcal{E} \cong \Sigma^{\mathbb{N}}. \tag{1}$$

The isomorphism $f : \mathcal{E} \to \Sigma^{\mathbb{N}}$ is given by

$$f(S) = (\exists n \in \mathbb{N} \,.\, n \in S).$$

The value $f(S)$ is in Σ because $n \in S$ is equivalent to the truth value $(\exists k \in \mathbb{N} \,.\, e(k) = n)$ where e is any enumeration of S. The inverse map $g : \Sigma^{\mathbb{N}} \to \mathcal{E}$ is defined as follows. Given any $p \in \Sigma^{\mathbb{N}}$, by the Axiom of Countable Choice there is $q : \mathbb{N} \times \mathbb{N} \to 2$ such that $p(n)$ is equivalent to $\exists k \in \mathbb{N} \,.\, q(n, k)$. We define $g(p)$ to be the countable set enumerated by the map $\mathbb{N} \times \mathbb{N} \to 1 + \mathbb{N}$, defined by

$$\langle n, k \rangle \mapsto \begin{cases} n & \text{if } q(n,k), \\ \star & \text{otherwise.} \end{cases}$$

The verification that f and g are inverses of each other is not terribly interesting. The upshot of (1) is that the Enumeration Axiom is just an instance of the precondition of Lawvere's theorem.

4. Applications of Lawvere's fixed-point theorem

So far we have not appealed to the Enumeration axiom, but now we will use it to transfer countability of $\mathcal{E}$ to countability of countably generated ωcppos, which will provide us with examples of Lawvere's fixed-point theorem. We need just one more lemma.

Lemma 4.1. *A countable inhabited poset $(P, \leq)$ with a decidable order contains a chain $c : \mathbb{N} \to P$ such that if P is linearly ordered then c is cofinal in P.*

Proof. Let $e : \mathbb{N} \to P$ be an enumeration of P. Define $c : \mathbb{N} \to P$ by

$$c_0 = e_0 \qquad \text{and} \qquad c_{n+1} = \begin{cases} e_{n+1} & \text{if } c_n \leq e_{n+1} \\ c_n & \text{otherwise} \end{cases}$$

We need to show that c is cofinal in P when P is linearly ordered. Clearly, c dominates e_0 because $e_0 = c_0$. Given any $n \in \mathbb{N}$, $c_n \leq e_{n+1}$ or $e_{n+1} \leq c_n$ since P is a linear order. In the former case we have $e_{n+1} = c_{n+1}$, and in the latter $e_{n+1} \leq c_n$. □

Theorem 4.2. *A countably generated* ω*cppo is countable.*

Proof. Let $(P, \leq)$ be an ωcppo and $b : \mathbb{N} \to P$ an enumeration of the basic elements. For $n \in \mathbb{N}$, the sub-poset

$$P_n = \{\bot\} \cup \{b_i \mid i \in \mathsf{W}_n\} \subseteq P$$

is countable and inhabited, and it has decidable order because b enumerates a base. We may apply Lemma 4.1 and the Axiom of countable Choice to obtain for each $n \in \mathbb{N}$ a countable chain $c^{(n)} : \mathbb{N} \to P_n$ such that $c^{(n)}$ is cofinal in P_n if P_n is a linear order. We claim that the map $e : \mathbb{N} \to P$ defined by $e(n) = \bigvee_k c_k^{(n)}$ is a surjection. Given any $x \in P$ there is a countable chain $d : \mathbb{N} \to P$ of basic elements whose supremum is x. Without loss of generality we may assume $d_0 = \bot$. By the Axiom of Countable Choice there is $r : \mathbb{N} \to \mathbb{N}$ such that $d = b \circ r$. By the Enumeration axiom there is $m \in \mathbb{N}$ such that $\mathsf{W}_m = \{r_i \mid i \in \mathbb{N}\}$. Notice that P_m is the image of d:

$$P_m = \{\bot\} \cup \{b_j \mid j \in \mathsf{W}_m\} = \{\bot\} \cup \{b_{r(i)} \mid i \in \mathbb{N}\} = \{d_j \mid j \in \mathbb{N}\}.$$

Therefore, P_m is linearly ordered, $c^{(m)}$ is a cofinal chain in P_m, and so

$$e(m) = \bigvee_k c_k^{(n)} = \bigvee_j d_j = x. \qquad \square$$

If $(P, \leq)$ is a countably generated ωcppo then by Theorem 3.2 so is $P^{\mathbb{N}}$. Theorem 4.2 applied to $P^{\mathbb{N}}$ provides a surjection $\mathbb{N} \to P^{\mathbb{N}}$, after which we may apply Lawvere's theorem. An immediate consequence is the fixed point property of countably generated ωcppos.

Corollary 4.3. *Every endomap on a countably generated* ω*cppo has a fixed point.*

Because Σ has the fixed point property but 2 and Ω do not (consider negation), there is a chain of proper inclusions

$$2 \subsetneq \Sigma \subsetneq \Omega.$$

Consequently Excluded middle and the Limited Principle of Omniscience are both false because they respectively state $2 = \Omega$ and $2 = \Sigma$.

The fixed point property implies a general version of Rice's theorem:

Corollary 4.4. *A decidable predicate on a countably generated* ω*cppo is trivial.*

Proof. A decidable predicate on a countably generated ωcppo $(P, \leq)$ is classified by a map $P \to 2$, which is constant by Proposition 2.4. □

4.1. **Richman's Axiom and Church's Thesis.** By Theorem 4.2 there is an enumeration

$$\varphi : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$$

of partial maps with countable graphs. Richman [17] showed how one may develop basic computability theory axiomatically from the existence of φ in the context of Bishop's constructive mathematics [2]. The work was taken further by Bridges and Richman [3].

Richman's axiom and the Enumeration axiom imply each other. We have already established one direction, while for the other all one has to notice is that there is a retraction $r : {\mathbb{N}_\bot}^{\mathbb{N}} \to \mathcal{E}$ and a section $s : \mathcal{E} \to {\mathbb{N}_\bot}^{\mathbb{N}}$, namely

$$r(f) = \{n \in \mathbb{N} \mid 0 \in f(n)\} \qquad \text{and} \qquad s(S)(n) = \{0 \mid n \in S\}.$$

Computability theory in logical form may also be developed from the formal Church's thesis [23, §1.11.7]

$$\forall f \in \mathbb{N}^{\mathbb{N}} .\, \exists k \in \mathbb{N} .\, \forall n \in \mathbb{N} .\, \exists m \in \mathbb{N} .\, T(k, n, m) \wedge U(m) = f(n).$$

Here $T(k, n, m)$ is Kleene's predicate expressing the fact that m encodes the computation of the Turing machine encoded by k with input n, and U extracts the output from the code of a computation. Thus the above statement says that every total function f is computed by some Turing machine.

Proposition 4.5. *The formal Church's thesis implies the Enumeration axiom.*

Proof. Define $W : \mathbb{N} \to \mathcal{E}$ by

$$W(k) = \{n \in \mathbb{N} \mid \exists \ell, m \in \mathbb{N} .\, T(k, \langle n, \ell \rangle, m) \wedge U(m) = 1\}.$$

We claim that W is a surjection if formal Church's thesis holds. For this purpose, consider an enumeration $e : \mathbb{N} \to 1 + S$ of some $S \in \mathcal{E}$. Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(\langle n, \ell \rangle) = \begin{cases} 1 & \text{if } \exists i < \ell .\, e(i) = n, \\ 0 & \text{otherwise.} \end{cases}$$

and observe that $S = \{n \in \mathbb{N} \mid \exists \ell \in \mathbb{N} .\, f(\langle n, \ell \rangle) = 1\}$. By formal Church's thesis there is a code k for f, and so

$$\begin{aligned} W(k) = \{n \in \mathbb{N} \mid \exists \ell, m \in \mathbb{N} .\, T(k, \langle n, \ell \rangle, m) \wedge U(m) = 1\} \\ = \{n \in \mathbb{N} \mid f(\langle n, \ell \rangle) = 1\} = S. \quad \square \end{aligned}$$

We cannot reverse Proposition 4.5, because the Enumeration axiom is valid in a realizability topos over Turing machines with a (fixed) oracle, and in such a topos not every map is Turing computable. The Enumeration axiom still implies a kind of 'synthetic' formal Church's thesis stating that every map $\mathbb{N} \to \mathbb{N}$ has a *synthetic* code, in the sense that $\forall f \in \mathbb{N} .\, \exists k \in \mathbb{N} .\, \varphi_k = f$. However, this says little beyond the fact that $\mathbb{N}^{\mathbb{N}}$ is *sub*countable because it is a subset of the countable set ${\mathbb{N}_\bot}^{\mathbb{N}}$.

Richman [17] notes that *"the main results of Church-Markov-Turing theory of computable functions may quickly be derived and understood without recourse to the largely irrelevant theories of recursive functions, Markov algorithms, or Turing machines."* While I hesitate to call central notions of computability theory irrelevant, the present paper does reinforce Richman's observation by developing an even larger portion of basic computability theory in an abstract way and without reference to any notion of computation.

5. The recursion theorem

There is another fixed point theorem that applies to countably generated ωcppos, and in fact to all ωcppos, namely the Knaster–Tarski theorem [10, 20].

Theorem 5.1. *Every continuous endomap on an ωcppo has the least fixed point.*

Proof. The usual proof is constructive. The least fixed point of a continuous endomap $f : P \to P$ on an ωcppo $(P, \leq)$ is computed as the supremum of the chain

$$\bot \leq f(\bot) \leq f(f(\bot)) \leq f(f(f(\bot))) \leq \cdots$$

of iterates of f applied to $\bot$. □

Theorem 5.1 applies only to *continuous* endomaps but gives canonical fixed points, whereas Corollary 4.3 applies to *all* maps but gives arbitrary fixed points. To get the best of both theorems, we will prove Theorem 6.4 stating that all functions from ωcpos to domains are continuous, and thus all endomaps on domains have least fixed points. The proof relies on a synthetic Recursion Theorem, which is the topic of the present section. The theorem takes the form of Lawvere's fixed-point theorem for multi-valued maps.

Let $\mathcal{P}_*(A)$ be the set of inhabited subsets of A,

$$\mathcal{P}_*(A) = \{S \in \Omega^A \mid \exists x \in A \,.\, x \in S\}.$$

A *multi-valued* map $f : A \rightrightarrows B$ is a map $f : A \to \mathcal{P}_*(B)$. A fixed point of a multi-valued map $f : A \to \mathcal{P}_*(A)$ is an element $x \in A$ such that $x \in f(x)$.

Theorem 5.2 (Recursion Theorem)**.** *If there is a surjection $e : \mathbb{N} \to A^{\mathbb{N}}$ then every multi-valued map $f : A \rightrightarrows A$ has a fixed point, which is an $x \in A$ such that $x \in f(x)$.*

Proof. For every $n \in \mathbb{N}$ there is $x \in f(e(n)(n))$, hence by the Axiom of Countable Choice there is a map $g : \mathbb{N} \to A$ such that $g(n) \in f(e(n)(n))$ for all $n \in \mathbb{N}$. There is $k \in \mathbb{N}$ such that $e(k) = g$, and so $e(k)(k) = g(k) \in f(e(k)(k))$. □

Note that the Recursion Theorem applies to countably generated ωcppos by Theorems 4.2 and 3.2. We shall use it only once in Proposition 6.1, where it is applied to Σ. (If one looked for shortcuts, one would point out that the Enumeration Axiom directly validates the application of the Recursion Theorem to Σ.)

The usual Kleene-Rogers Recursion Theorem [8, 9, 18] is a consequence of the synthetic one. Recall from §3.1 the enumeration $\varphi : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$ of the domain ${\mathbb{N}_\bot}^{\mathbb{N}}$.

Corollary 5.3. *Given any map $f : \mathbb{N} \to \mathbb{N}$, there is $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{f(n)}$.*

Proof. Recursion Theorem applies to the domain ${\mathbb{N}_\bot}^{\mathbb{N}}$ because $({\mathbb{N}_\bot}^{\mathbb{N}})^{\mathbb{N}}$ is is countable by a double application of Theorem 3.2. Take a fixed point g of the multi-valued map $F : {\mathbb{N}_\bot}^{\mathbb{N}} \rightrightarrows {\mathbb{N}_\bot}^{\mathbb{N}}$ defined by

$$F(g) = \{h \in {\mathbb{N}_\bot}^{\mathbb{N}} \mid \exists n \in \mathbb{N} \,.\, \varphi_n = g \wedge h = \varphi_{f(n)}\}.$$

By the definition of F there is $n \in \mathbb{N}$ such that $\varphi_n = g = \varphi_{f(n)}$. □

The technique used to prove the corollary may be generalized as follows.

Proposition 5.4. *Suppose $e : A \to B$ is a surjection and every multi-valued map on B has a fixed point. Then for every $h : A \rightrightarrows B$ there exists $x \in A$ such that $e(x) \in h(x)$.*

Proof. Consider the multi-valued map $F : B \rightrightarrows B$ defined by

$$F(y) = \{z \in B \mid \exists x \in A \,.\, e(x) = y \wedge z \in h(x)\}.$$

There is $y \in B$ such that $y \in F(y)$, which implies that for some $x \in A$ we have $e(x) = y \in h(x)$. □

To recover Corollary 5.3 from the Proposition 5.4, take the surjection $\varphi : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$ and the map $h = \varphi \circ f : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$. Various self-referential objects can be constructed with the help of Proposition 5.4, too. For example, to obtain $n \in \mathbb{N}$ such that $\mathsf{W}_n = \{n\}$, take the surjection $\mathsf{W} : \mathbb{N} \to \mathcal{E}$ and the map $h : \mathbb{N} \to \mathcal{E}$ defined by $h(k) = \{k\}$. To obtain a quine (a function that outputs its own code) take the surjection $\varphi : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$ and the map $h : \mathbb{N} \to {\mathbb{N}_\bot}^{\mathbb{N}}$ defined be $h(n) = (\lambda k \in \mathbb{N} \,.\, n)$.

6. A continuity principle

In the present section we prove a continuity principle for maps from ωcpos to domains. We first show that the Enumeration axiom and Markov's principle together imply that maps into Σ satisfy a weak form of sequential continuity stating that the limit point of a sequence cannot be detached from the sequence. For maps from ωcpos to Σ the weak principle may be amplified to proper continuity, from which the general theorem then follows.

The set of non-increasing binary sequences

$$\mathbb{N}^+ = \{\alpha \in 2^{\mathbb{N}} \mid \forall n \in \mathbb{N} \,.\, \alpha_n \geq \alpha_{n+1}\}.$$

can be thought of as a generic sequence with a limit point, where the n-th term is

$$\overline{n} = \underbrace{1, 1, \ldots, 1}_{n} 0, 0, \ldots$$

and the limit point is

$$\infty = 1, 1, 1, 1, \ldots$$

We define a strict order $<$ on $\mathbb{N}^+$ by

$$\alpha < \beta \iff \exists n \in \mathbb{N} \,.\, \alpha_n < \beta_n.$$

Recall from [1] the principle of *Weak Sequential Openness (WSO)*:

$$\forall U : \mathbb{N}^+ \to \Sigma \,.\, (U(\infty) = \top \Rightarrow \exists n \in \mathbb{N} \,.\, U(\overline{n}) = \top).$$

The principle is motivated by considerations from *synthetic topology*, see [1, 13], where Σ^A is seen as the topology of *intrinsically open* subsets in A. From a topological perspective the principle claims that the limit point ∞ cannot be isolated from the sequence $\overline{0}, \overline{1}, \overline{2}, \ldots$

Proposition 6.1. *Weak Sequential Openness holds.*

Proof. Suppose $U : \mathbb{N}^+ \to \Sigma$ is such that $U(\infty) = \top$. Define $g : \Sigma \rightrightarrows \Sigma$ by

$$g(p) = \{q \in \Sigma \mid \exists \alpha \in \mathbb{N}^+ \,.\, (\alpha < \infty) = p \land q = U(\alpha)\}.$$

By Recursion Theorem there is $p \in \Sigma$ such that $p \in g(p)$, and by definition of g there is $\alpha \in \mathbb{N}^+$ such that $(\alpha < \infty) = p = U(\alpha)$. If it were the case that $p = \bot$ then it would follow that $\alpha = \infty$, but this would lead to the contradiction $\bot = U(\infty) = \top$. Therefore $p \neq \bot$ and so $p = \top$ by Markov's principle, expressed in the form given in Proposition 3.4. Because $\alpha < \infty$ there is $n \in \mathbb{N}$ such that $\alpha = \overline{n}$, hence $U(\overline{n}) = \top$. □

We can use the principle to show that maps from ωcpos into Σ are well behaved.

Lemma 6.2. *Every map $f : P \to \Sigma$ on an ωcpo $(P, \leq)$ is monotone.*

Proof. Consider any $u, v \in P$ such that $u \leq v$. We need to show that $f(u) = \top$ implies $f(v) = \top$, so assume $f(u) = \top$. Define $r : \mathbb{N}^+ \times \mathbb{N} \to P$ and $s : \mathbb{N}^+ \to P$ by

$$r(\alpha, n) = \begin{cases} u & \text{if } \alpha_n = 1, \\ v & \text{if } \alpha_n = 0, \end{cases} \qquad \text{and} \qquad s(\alpha) = \textstyle\bigvee_n r(\alpha, n).$$

Observe that $s(\infty) = u$ and $s(\overline{n}) = v$ for $n \in \mathbb{N}$. Because $f(s(\infty)) = f(u) = \top$, by Weak Sequential Openness there is $n \in \mathbb{N}$ such that $f(s(\overline{n})) = \top$, hence $f(v) = f(s(\overline{n})) = \top$. □

Lemma 6.3. *Every map $f : P \to \Sigma$ on an ωcpo $(P, \leq)$ is continuous.*

Proof. Let $c : \mathbb{N} \to P$ be any chain. By the previous lemma f is monotone, therefore $\bigvee_n f(c_n) \leq f(\bigvee_n c_n)$ holds. To establish the opposite inequality it suffices to show that $f(\bigvee_n c_n) = \top$ implies $f(c_n) = \top$ for some $n \in \mathbb{N}$, so assume $f(\bigvee_n c_n) = \top$.

For $\alpha \in \mathbb{N}^+$, define the chain $c^\alpha : \mathbb{N} \to P$ by

$$c^\alpha_n = \begin{cases} c_n & \text{if } \alpha_n = 1, \\ c_k & \text{if } \alpha = \overline{k} \text{ and } k < n. \end{cases}$$

Observe that $c^\infty = c$ and that $c^{\overline{n}}$ increases up to c_n and stays put there. Define $s : \mathbb{N}^+ \to P$ by $s(\alpha) = \bigvee_n c^\alpha_n$. Because $f(s(\infty)) = f(\bigvee_n c_n) = \top$, by Weak Sequential Openness there is $n \in \mathbb{N}$ such that $f(s(\overline{n})) = \top$, hence $f(c_n) = f(s(\overline{n})) = \top$, as desired. □

Everything is set in place for a continuity principle.

Theorem 6.4. *Every map $f : P \to Q$ from an ωcpo $(P, \leq_P)$ to a domain $(Q, \leq_Q)$ is continuous.*

Proof. If $x \in Q$ is a basic element and $y \in Q$ then $(x \leq_Q y) \in \Sigma$. Indeed, $y = \bigvee_n d_n$ for a chain of basic elements $d : \mathbb{N} \to Q$, so by compactness of x

$$x \leq_Q y \iff \exists n \in \mathbb{N} \,.\, x \leq_Q d_n.$$

The right-hand side is a truth value in Σ because the order $\leq_Q$ is decidable on basic elements. Therefore, for a compact $x \in Q$ we may define a map $u_x : Q \to \Sigma$ by

$$u_x(y) = (x \leq_Q y).$$

Let $c : \mathbb{N} \to P$ be a chain. In order to prove $f(\bigvee_n c_n) = \bigvee_n f(c_n)$ it suffices to show that, for every basic element $x \in Q$,

$$x \leq f(\textstyle\bigvee_n c_n) \iff x \leq \bigvee_n f(c_n).$$

Another way of saying the same thing is

$$u_x(f(\textstyle\bigvee_n c_n)) = u_x(\bigvee_n f(c_n)),$$

which holds because by Lemma 6.3 both $u_x \circ f$ and u_x are continuous, therefore

$$u_x(f(\textstyle\bigvee_n c_n)) = (u_x \circ f)(\bigvee_n c_n) = \bigvee_n u_x(f(c_n)) = u_x(\bigvee_n f(c_n)).$$ □

Theorem 6.4 subsumes several other continuity principles. When we instantiate it to endomaps ${\mathbb{N}_\bot}^{\mathbb{N}} \to {\mathbb{N}_\bot}^{\mathbb{N}}$ we obtain a synthetic version of the Myhill–Shepherdson theorem [15], while the instance $\Sigma^{\mathbb{N}} \to \Sigma$ corresponds to Scott's principle from synthetic domain theory [21].

At last, let us reconcile Lawvere's and Tarski-Knaster fixed point theorems.

Corollary 6.5. *Every endomap on a domain has a least fixed point.*

Proof. The Knaster–Tarski Theorem 5.1 applies because such an endomap is continuous. □

REFERENCES

[1] A. Bauer and D. Lešnik. Metric spaces in synthetic topology. *Annals of pure and applied logic*, 163(2):87–100, 2012.

[2] E. Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, 1967.

[3] D. Bridges and F. Richman. *Varieties of Constructive Mathematics*. Number 97 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1987.

[4] Y. L. Ershov. *The theory of enumerations*. Nauka, 1980.

[5] M. Escardó. Synthetic topology of data types and classical spaces. *Electronic Notes in Theoretical Computer Science*, 87:21–156, 2004.

[6] J. M. E. Hyland. The effective topos. In A.S. Troelstra and D. Van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North Holland Publishing Company, 1982.

[7] J. M. E. Hyland. First steps in synthetic domain theory. In *Category Theory*, number 1488 in Lecture Notes in Mathematics, 1991.

[8] S. C. Kleene. On notation for ordinal numbers. *Journal of Symbolic Logic*, 3(4):150–155, 1938.

[9] S. C. Kleene. *Introduction to Metamathematics*. North-Holland, 1952.

[10] B. Knaster. Un théorème sur les fonctions d'ensembles. *Annales Polonici Mathematici*, 6:133–134, 1928.

[11] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[12] F. W. Lawvere. Diagonal arguments and cartesian closed categories. *Lecture Notes in Mathematics*, 92:134–145, 1969. Republished in: Reprints in Theory and Applications of Categories, No. 15 (2006), 1–13.

[13] D. Lešnik. *Synthetic Topology and Constructive Metric Spaces*. PhD thesis, University of Ljubljana, 2010.

[14] A. A. Markov. On the continuity of constructive functions. *Uspekhi Matematicheskikh Nauk*, 9(3/61):226–230, 1954.

[15] J. Myhill and J. C. Shepherdson. Effective operations on partial recursive functions. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1(4):310–317, 1955.

[16] W. Phoa. *Domain Theory in Realizability Toposes*. PhD thesis, Trinity College, Cambridge, 1991.

[17] F. Richman. Church's thesis without tears. *The Journal of Symbolic Logic*, 48(3):797–803, September 1983.

[18] H. Rogers. *Theory of Recursive Functions and Effective Computability*. MIT Press, 1967.

[19] G. Rosolini. *Continuity and Effectiveness in Topoi*. PhD thesis, University of Oxford, 1986.

[20] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.

[21] P. Taylor. The fixed point property in synthetic domain theory. In *Proceedings of the Sixth Annual Symposium on Logic in Computer Science (LICS '91), Amsterdam, The Netherlands, July 15-18, 1991*, pages 152–160, 1991.

[22] P. Taylor. Foundations for computable topology. In G. Sommaruga, editor, *Foundational Theories of Classical and Constructive Mathematics*, volume 76 of *The Western Ontario Series in Philosophy of Science*, pages 265–310. Springer Netherlands, 2011.

[23] A. S. Troelstra. *Metamathematical investigation of intuitionistic arithmetic and analysis*, volume 344 of *Lecture notes in mathematics*. Springer, 1973.

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