First Steps in Synthetic Computability

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  - surprising theorems
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- Can we do computability theory as "ordinary" math?
  - use axiomatic method
  - argue conceptually and abstractly
  - use customary mathematical notions

Related Work

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- Hyland [1982], effective topos
- Richman [1984], an axiom for effective enumerability of partial functions
- We shall follow Richman [1984] in style, and borrow ideas from Rosolini [1986], Berger [1983], and Spreen [1998].

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- Add a couple of axioms about sets of numbers.
- The underlying logic is *intuitionistic*:
  this is a theorem, not a political conviction.
- Interpretation in the effective topos translates our theory back to classical recursion theory.

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  generally, no Law of Excluded Middle or Proof by Contradiction.

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- We say that A is
  - *non-empty* if $\neg \forall x \in A . \bot$,
  - *inhabited* if $\exists x \in A . \top$.

Some interesting sets

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- $2 \subseteq \Omega_{\neg\neg} \subseteq \Omega$.

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The generic convergent sequence

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$$\mathbb{N}^+ = \left\{a \in 2^{\mathbb{N}} \mid \forall k \in \mathbb{N} \,.\, a_k \leq a_{k+1}\right\} \,.$$

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- But there is also $\infty = \lambda k.\,0$.
- $\mathbb{N}^+$ can be thought of as the one-point compactification of $\mathbb{N}$.

Enumerable & finite sets

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- A is *enumerable (countable)* if there exists an onto map $e : \mathbb{N} \twoheadrightarrow \mathtt{1} + A$, called an *enumeration* of A. For inhabited A we may take $e : \mathbb{N} \twoheadrightarrow A$.

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- A is *infinite* if there exists an injective $a : \mathbb{N} \rightarrowtail A$.

Lawvere → Cantor

Theorem (Lawvere)

If $e : A \to B^A$ is onto then B has the fixed point property.

Proof.

Given $f : B \to B$, there is $x \in A$ such that
$e(x) = \lambda y : A \,.\, f(e(y)(y))$. Then $e(x)(x) = f(e(x)(x))$. □

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Corollary (Cantor)

There is no onto map $e : A \twoheadrightarrow \mathcal{P}A$.

Proof.

$\mathcal{P}A = \Omega^A$ and $\neg : \Omega \to \Omega$ does not have a fixed point. □

Non-enumerability of Cantor and Baire space

Corollary

$2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are not enumerable.*

Proof.

2 and $\mathbb{N}$ do not have the fixed-point property. □

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We have proved our first synthetic theorem: there are no effective enumerations of recursive sets and total recursive functions.

Projection Theorem

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A subset of $\mathbb{N}$ is enumerable iff it is the projection of a decidable subset of $\mathbb{N} \times \mathbb{N}$.

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If A is the projection of $B \subseteq \mathbb{N} \times \mathbb{N}$, define $e : \mathbb{N} \times \mathbb{N} \to \mathtt{1} + A$ by

$$e\langle m, n\rangle = \mathtt{if}\ \langle m, n\rangle \in B\ \mathtt{then}\ m\ \mathtt{else}\ \star \ . \qquad \square$$

Semidecidable sets

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- The set of semidecidable truth values:

$$\Sigma = \left\{ p \in \Omega \mid \exists d \in 2^{\mathbb{N}} \, . \, p = \exists n \in \mathbb{N} \, . \, d(n) \right\} \, .$$

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- A subset $S \subseteq \mathbb{N}$ is *semidecidable* if $\chi_S : A \to \Sigma$.

Σ as a quotient of $\mathbb{N}^+$

- Σ is a quotient of $2^{\mathbb{N}}$ via taking countable joins: $d \in 2^{\mathbb{N}}$ is mapped to $\exists\, n \in \mathbb{N} \,.\, d(n)$.

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- Σ is a quotient of $\mathbb{N}^+$ via the map $q : \mathbb{N}^+ \to \Sigma$, defined by $q(t) = (t < \infty)$.
- If $q(t) = s$ we say that t is a *time at which s becomes true*. Beware, t is not uniquely determined!

Semidecidable subsets

Theorem

The enumerable subsets of $\mathbb{N}$ *are precisely the semidecidable subsets of* $\mathbb{N}$.

Proof.

By Projection Theorem, an enumerable $A \subseteq \mathbb{N}$ is the projection of a decidable $B \subseteq \mathbb{N} \times \mathbb{N}$. Then $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, \langle n, m \rangle \in B$.
Conversely, if $A \in \Sigma^{\mathbb{N}}$, by Number Choice there is $d : \mathbb{N} \times \mathbb{N} \to 2$ such that $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, d(m, n)$. □

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- The *topology of A* is Σ^A.

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- For $s \in \widetilde{B}$, write $s\downarrow$ when s is inhabited.
- Which partial functions $\mathbb{N} \to \widetilde{\mathbb{N}}$ have enumerable graphs?

Σ-partial functions

Proposition

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Define the *lifting* operation

$$A_\perp = \left\{ s \in \widetilde{A} \mid s{\downarrow} \in \Sigma \right\} .$$

For $f : A \to B$ define $f_\perp : A_\perp \to B_\perp$ to be

$$f_\perp(s) = \{ f(x) \mid x \in s \} .$$

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$$f_\perp(s) = \{ f(x) \mid x \in s \} .$$

A Σ*-partial function* is a function $f : A \to B_\perp$.

Domains of Σ-partial functions

Proposition

A subset is semidecidable iff it is the domain of a Σ*-partial function.*

Proof.

A semidecidable subset $S \in \Sigma^A$ is the domain of its characteristic map $\chi_S : A \to \Sigma = \mathbf{1}_\perp$.
If $f : A \to B_\perp$ is Σ-partial then its domain is the set $\{x \in A \mid f(x)\downarrow\}$, which is obviously semidecidable. □

The Single-Value Theorem

A *selection* for $R \subseteq A \times B$ is a partial map $f : A \rightharpoonup B$ such that, for every $x \in A$,

$$\exists\, y \in B \,.\, R(x, y) \implies f(x){\downarrow} \land R(x, f(x)) \,.$$

This is like a choice function, expect it only chooses when there is something to choose from.

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Theorem (Single Value)

Every open relation $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$ has a Σ-partial selection.

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Proposition

Σ *and* $\mathcal{E}$ *have the fixed-point property.*

Proof.

By Lawvere, $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E} = \Sigma^{\mathbb{N}} \cong \Sigma^{\mathbb{N} \times \mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

Enumerability of $\mathbb{N} \to \mathbb{N}_\bot$

Proposition

$\mathbb{N} \to \mathbb{N}_\bot$ *is enumerable.*

Proof.

Let $V : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$ be an enumeration. By Single-Value Theorem and Number Choice, there is $\varphi : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\bot)$ such that φ_n is a selection of V_n. The map φ is onto, as every $f : \mathbb{N} \to \mathbb{N}_\bot$ is the only selection of its graph. □

The Law of Excluded Middle Fails

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Corollary

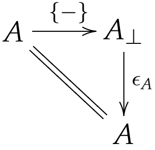
The Law of Excluded Middle is false.

Proof.

Among the sets $2 \subseteq \Sigma \subseteq \Omega$ only the middle one has the fixed-point property, so $2 \neq \Sigma \neq \Omega$. □

Focal sets

- A *focal set* is a set A together with a map $\epsilon_A : A_\perp \to A$ such that $\epsilon_A(\{x\}) = x$ for all $x \in A$:



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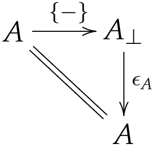

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- A lifted set $A_\perp$ is always focal (because lifting is a monad with whose unit is $\{-\}$).

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- A *flat domain* $A_\perp$ is focal. It is enumerable if A is decidable and enumerable.
- If A is enumerable and focal then so is $A^{\mathbb{N}}$:

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{e_\perp^{\mathbb{N}}} A_\perp^{\mathbb{N}} \xrightarrow{\epsilon_A^{\mathbb{N}}} A^{\mathbb{N}}$$

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- Some enumerable focal sets are

$$\Sigma^{\mathbb{N}}, \quad 2_\perp^{\mathbb{N}}, \quad \mathbb{N}_\perp^{\mathbb{N}} \; .$$

Multi-valued functions

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- A *fixed point* of $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Recursion Theorem

Theorem (Recursion Theorem)

Every $f : A \rightrightarrows A$ on enumerable focal A has a fixed point.

Proof.

Let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration, and $\epsilon : A_\perp \to A$ a focal map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $e(m) \in f(e(k))$. By Number Choice there is a map $c : \mathbb{N} \to \mathbb{N}$ such that $e(c(k)) \in f(e(k))$ for every $k \in \mathbb{N}$. It suffices to find k such that $e(c(k)) = e(k)$ since then $x = e(k)$ is a fixed point for f.

For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\epsilon(e_\perp(c_\perp(\varphi_m(m)))) = e(n)$. By Number Choice there is $g : \mathbb{N} \to \mathbb{N}$ such that $\epsilon(e_\perp(c_\perp(\varphi_m(m)))) = e(g(m))$ for every $m \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $g = \varphi_j$. Let $k = g(j)$. Then

$$e(k) = e(g(j)) = \epsilon(e_\perp(c_\perp(\varphi_j(j)))) == e(c(g(j))) = e(c(k)) \ .$$

□

Classical Recursion Theorem

Corollary (Classical Recursion Theorem)

For every $f : \mathbb{N} \to \mathbb{N}$ *there is* $n \in \mathbb{N}$ *such that* $\varphi_{f(n)} = \varphi_n$.

Proof.

In Recursion Theorem, take the enumerable focal set $\mathbb{N}_\perp^\mathbb{N}$ and the multi-valued function

$$F(g) = \left\{h \in \mathbb{N}_\perp^\mathbb{N} \mid \exists n \in \mathbb{N} \,.\, g = \varphi_n \wedge h = \varphi_{f(n)}\right\} .$$

There is g such that $g \in F(g)$. Thus there exists $n \in \mathbb{N}$ such that $\varphi_n = g = h = \varphi_{f(n)}$. □

Markov Principle

- If a binary sequence $a \in 2^{\mathbb{N}}$ is not constantly 0, does it contain a 1?

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Axiom (Markov Principle)

A binary sequence which is not constantly 0 *contains a* 1.

Post's Theorem

Theorem (Post)

A subset is decidable if, and only if, it and its complement are both semidecidable.

Proof.

Clearly, a decidable proposition is semidecidable and so is its complement. If p and $\neg p$ are semidecidable then so is $p \vee \neg p$. By Markov Principle $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, hence

$$p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top \,,$$

as required. □

Topological Exterior and Creative Sets

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- The *exterior* of an open set is the largest open set disjoint from it.
- An open set $U \in \Sigma^A$ is *creative* if it is without exterior: for every $V \in \Sigma^A$ such that $U \cap V = \emptyset$ there is $V' \in \Sigma^A$ such that $U \cap V' = \emptyset$ and $V' \setminus V$ is inhabited.

Theorem

There exists a creative subset of $\mathbb{N}$.

Proof.

The familiar $K = \{n \in \mathbb{N} \mid n \in \mathsf{W}_n\}$ is creative. Given any $V \in \mathcal{E}$ with $V = \mathsf{W}_k$ and $K \cap V = \emptyset$, we have $n \notin V$, so we can take $V' = V\,\{k\}$. □

Immune and Simple Sets

- A set is *immune* if it is neither finite nor infinite.

Immune and Simple Sets

- A set is *immune* if it is neither finite nor infinite.
- A set is *simple* if it is open and its complement is immune.

Theorem

There exists a closed subset of $\mathbb{N}$ which is neither finite nor infinite.

Proof.

Following Post, consider $P = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \land n \in W_m\}$, and let $f : \mathbb{N} \to \mathbb{N}_\perp$ be a selection for P by Single-Value Theorem. Then $S = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N} . f(m) = n\}$ is the complement of the set we are looking for.
Because $f(m) > 2m$ the set $\mathbb{N} \setminus S$ cannot be finite.
For any infinite enumerable set $U \subseteq \mathbb{N} \setminus S$ with $U = \mathsf{W}_m$, we have $f(m)\downarrow$, $f(m) \in \mathsf{W}_m = U$, and $f(m) \in S$, hence U is not contained in $\mathbb{N} \setminus S$. □

Inseparable sets

Theorem

There exists an element of Plotkin's $\mathbf{2}_\perp^\mathbb{N}$ that is inconsistent with every maximal element of $\mathbf{2}_\perp^\mathbb{N}$.

Proof.

Because $\mathbf{2}_\perp$ is focal and enumerable, $\mathbf{2}_\perp^\mathbb{N}$ is as well. Let $\psi : \mathbb{N} \twoheadrightarrow \mathbf{2}_\perp^\mathbb{N}$ be an enumeration, and let $t : \mathbf{2}_\perp \to \mathbf{2}_\perp$ be the isomorphism $t(x) = \neg_\perp x$ which exchanges 0 and 1. Consider $a \in \mathbf{2}_\perp^\mathbb{N}$ defined by $a(n) = t(\psi_n(n))$. If $b \in \mathbf{2}_\perp^\mathbb{N}$ is maximal with $b = \psi_k$, then $a(k) = \neg\psi_k(k) = \neg b(k)$. Because $a(k)$ and $b(k)$ are both total and different they are inconsistent. Hence a and b are inconsistent. □

Berger's Lemma

Lemma (Berger)

If $U : A \rightrightarrows \Sigma$ is a multi-valued open set, and $x : \mathbb{N}^+ \to A$ such that $U(x_\infty) = \{\top\}$ then there is $k \in \mathbb{N}$ for which $\top \in U(x_k)$.

Proof.

For every $y \in A$ there is $p \in \mathbb{N}^+$ such that $(p < \infty) \in U(y)$.
Consequently, for every $y \in A$ there is $z \in A$ such that

$$\exists p \in \mathbb{N}^+ . ((p < \infty) \in U(y) \land z = x_p) \ . \tag{1}$$

By Recursion Theorem there is $y = z$ satisfying (1). For such y, p is not equal to ∞ because $p = \infty$ implies $y = x_\infty$ and $\bot = (p < \infty) \in U(y) = U(x_\infty) = \{\top\}$, contradiction. By Markov Principle, $p \in \mathbb{N}$ so we have $x_p = y$ and $\top = (p < \infty) \in U(x_p)$, as required. □

ω-Chain Complete Posets

- An *ω-chain complete poset (ω-cpo)* is a poset in which enumerable chains have suprema.

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- An *ω-chain complete poset (ω-cpo)* is a poset in which enumerable chains have suprema.
- A *base* for an ω-cpo $(A, \leq)$ is an enumerable subset $S \subseteq A$ such that:
  - For all $x \in S, y \in A$, $(x \leq y) \in \Sigma$.
  - Every $x \in A$ is the supremum of a chain in S.

The Topology of ω-cpos

Theorem

1. *The open subsets of an ω-cpo are upward closed and inaccessible by chains.*
2. *If an ω-cpo A has a base S, then every open is a union of basic opens sets $\uparrow x = \{y \in A \mid x \leq y\}$ with $x \in S$.*

Proof.

If $x \leq y$ and $x \in U \in \Sigma^A$, define $a : \mathbb{N}^+ \to A$ by

$$a_p = \bigcup_{k \in \mathbb{N}} \texttt{if}\ k < p\ \texttt{then}\ x\ \texttt{else}\ y$$

Then $a_\infty = x \in U$ and by Berger's Lemma there is $k \in \mathbb{N}$ such that $y = a_k \in U$, too. □

The Injectivity Axiom

A subset $A \subseteq B$ is a *subspace* if every $U \in \Sigma^A$ is the restriction of some $V \in \Sigma^B$.

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Axiom (Injectivity)

A classical subset of $\mathbb{N}$ *is a subspace of* $\mathbb{N}$.

In other words, Σ is injective with respect to classical subsets of $\mathbb{N}$.

Kreisel-Lacombe-Shoenfield Theorem

Theorem (Kreisel-Lacombe-Shoenfield-Ceitin)

Every map from a complete separable metric space to a metric space is $\epsilon\delta$-continuous.

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Every map from a complete separable metric space to a metric space is $\epsilon\delta$-continuous.

Proof idea.

Suppose $f : M \to L$ is such a function. Write $B(x, r)$ for the open ball with radius r and centered at x.
The proof uses Berger's Lemma and the observation that

$$\forall\, t \in B(x, r) \,.\, f(t) \in B(y, q)$$

is the negation of

$$\exists\, t \in B(x, r) \,.\, d(f(t), y) > q \;,$$

which is semidecidable. □

Where to go from here?

- Computable Analysis:
  - $2^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$,
  - $\mathbb{R}$ is locally non-compact, in the sense that every interval contains a sequence without accumulation point,
  - $\mathbb{R}$ has measure zero: it can be covered by a sequence of open intervals whose *total* length is bounded by $\epsilon > 0$.

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- Turing degrees:
  - find a connection between Turing degrees and Baire category theorems.

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- What do we learn from this?