On the Bourbaki-Witt Principle in Toposes

Andrej Bauer
University of Ljubljana, Slovenia
`Andrej.Bauer@andrej.com`

Peter LeFanu Lumsdaine
Dalhousie University, Halifax, Canada
`p.l.lumsdaine@mathstat.dal.ca`

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Abstract

The Bourbaki-Witt principle states that any progressive map on a chain-complete poset has a fixed point above every point. It is provable classically, but not intuitionistically.

We study this and related principles in an intuitionistic setting. Among other things, we show that Bourbaki-Witt fails exactly when the trichotomous ordinals form a set, but does not imply that fixed points can always be found by transfinite iteration. Meanwhile, on the side of models, we see that the principle fails in realisability toposes, and does not hold in the free topos, but does hold in all cocomplete toposes.

1 Introduction

The Bourbaki-Witt theorem [3, 15] states that a progressive map $f : P \to P$ on a chain-complete poset P has a fixed point above every point. (A map is *progressive* if $x \leq f(x)$ for all $x \in P$.) A classical proof of the Bourbaki-Witt theorem constructs the increasing sequence

$$x \leq f(x) \leq f^2(x) \leq \cdots \leq f^{\omega}(x) \leq f^{\omega+1}(x) \leq \cdots$$

where chain-completeness is used at limit stages. If the sequence is indexed by a large enough ordinal, it must stabilise, giving a fixed point of f above x.

It has been observed recently by the first author [2] that in the effective topos there is a counterexample to the Bourbaki-Witt theorem, as well as to the related Knaster-Tarski theorem. An earlier result of Rosolini [11] exhibits a model of intuitionistic set theory in which the (trichotomous) ordinals form a set, and since the successor operation has no fixed points, this also provides a counterexample to intuitionistic validity of the Bourbaki-Witt theorem.

The counterexamples bury any hope for an intuitionistic proof of the Bourbaki-Witt theorem. However, several questions still remain. Is the theorem valid in other toposes? How is it linked with the existence of large enough ordinals? How does it compare to Knaster-Tarski and other related fixed-point principles? We address these questions in the present paper.

1.1 Overview

After laying out the setting in Section 2, we begin in Subsection 3.1 by summarising the relationships between various fixed-point principles of the same form as the Bourbaki-Witt principle. In Subsection 3.2, we discuss several classically equivalent formulations of the Bourbaki-Witt principle, which turn out to be intuitionistically equivalent as well. Likewise, several ways of stating that the Bourbaki-Witt theorem fails are intuitionistically equivalent. In Subsection 3.3, we investigate the connection between the Bourbaki-Witt principle and iteration along ordinals, and prove that failure of the principle is equivalent to the trichotomous ordinals forming a set.

In Section 4, we change tack and investigate validity of the Bourbaki-Witt principle in various toposes. First we show that realisability toposes contain counterexamples to the principle. From this we conclude that the principle cannot hold in the free topos, as there is a definable chain-complete poset with a definable progressive map which is interpreted as a counterexample in the effective topos. Next we show that the Bourbaki-Witt principle transfers along geometric morphisms, and hence its validity in the category of classical sets implies validity in cocomplete toposes, so in particular in Grothendieck toposes. Finally, we show by topos-theoretic means that while the Bourbaki-Witt principle does imply that the ordinals cannot form a set, it does not imply that fixed-points can always be found by iteration along ordinals, as they can classically.

2 Preliminaries

The content of this paper takes place in two different logical settings. In the first setting we put on our constructive hats and prove theorems in intutionistic mathematics. Our proofs are written informally but rigorously in the style of Errett Bishop (but without countable choice). They can be interpreted in any elementary topos with natural numbers object [6, 8], or in an intuitionistic set theory such as IZF [1]. Since unbounded quantification is not available in topos logic, statements referring to all structures of a certain kind are to be interpreted as schemata, as is usual in that setting. When we meet a statement with an inner unbounded quantifer, we discuss it explicitly. Intuitionistic set theories do not suffer from this complication.

In the second setting we put on our categorical logicians' hats and prove meta-theorems about provability statements and topos models. In these arguments we use classical reasoning when necessary, including for Subsection 4.1 the Axiom of Choice.

Let us recall some basic notions and terminology. If P is a poset, a *chain* in P is a subset $C \subseteq P$ such that for all $x, y \in C$, $x \leq y$ or $y \leq x$. The set of chains in P is denoted by $\mathrm{Ch}(P)$. A subset $D \subseteq P$ is *directed* when every finite subset of D, including the empty set, has an upper bound in D; equivalently, if D is inhabited and every two elements in D have a common upper bound in D.

A poset P is *chain-complete* if every chain in P has a supremum, and is *directed-complete* if every directed subset of P has a supremum. Any chain-complete poset is inhabited by the supremum of the empty chain, whereas a directed-complete poset may be empty. However, any directed-complete poset with a bottom element is chain-complete: if C is a chain, then $C \cup \{\bot\}$ is directed, and its supremum gives a supremum for C. Since suprema are unique when they exist, a poset is chain-complete precisely when it has a supremum operator $\sup : \mathrm{Ch}(P) \to P$.

An endofunction $f : P \to P$ is called *progressive* (sometimes *inflationary* or *increasing*) if $x \leq f(x)$ for every $x \in P$. A point $x \in P$ is *fixed* by f if $f(x) = x$, *pre-fixed* if $f(x) \leq x$, and *post-fixed* if $x \leq f(x)$.

The *Bourbaki-Witt principle* is the statement

> *"A progressive map on a chain-complete poset has a fixed point above every point."*

3 Bourbaki-Witt in the constructive setting

3.1 Related fixed-point principles

The Bourbaki-Witt principle is one of a family of fixed-point principles, obtained by combining either progressive or monotone maps with either complete, directed-complete, or chain-complete posets. Three of the six combinations can be proved intuitionistically, as follows.

Theorem 3.1 (Tarski [12]) *Any monotone map on a complete lattice has a fixed point above every post-fixed point.*

Proof. Let $f : P \to P$ be such a map and $x \in P$ a post-fixed point, i.e., $x \leq f(x)$. Consider the set $S = \{y \in P \mid x \leq y \text{ and } f(y) \leq y\}$ of pre-fixed points above x. The infimum $z = \inf S$ is a pre-fixed point because by monotonicity $f(z) \leq f(y) \leq y$ for all $y \in S$. But also $x \leq z$, so z and $f(z)$ are in S, hence z is a post-fixed point as well. Thus z is a fixed point of f above x, and indeed by construction the least such. □

The usual formulation of Tarski's theorem states just that every monotone map has a fixed point; here we reformulate it to make it more similar to the Bourbaki-Witt theorem, but the two versions are equivalent.

Theorem 3.2 (Pataraia [9]) *Any monotone map on a directed-complete poset has a fixed point above every post-fixed point.*

Proof. We summarise the proof as given by Dacar [5]. Given a monotone $f : P \to P$ on a directed-complete poset P, let $Q = \{x \in P \mid x \leq f(x)\}$ be the subposet of post-fixed points. The set

$$M = \{g : Q \to Q \mid g \text{ is monotone and progressive}\}$$

contains the restriction of f to Q, is directed-complete under the pointwise ordering, and is itself directed: it contains the identity, and for any $g, h \in M$, the composite $g \circ h$ gives an upper bound of g and h. Thus M has a top element t, which must satisfy $g \circ t = t$ for all $g \in M$, hence $t(x)$ is a fixed point of f above x for any $x \in Q$. □

The third theorem which can be proved intuitionistically combines progressive maps and complete lattices, but it is completely trivial as the top element is always a fixed point of a progressive map. One might be tempted to save the theorem by proving that a progressive map on a complete lattice has a *least* fixed point, until one is shown a counterexample.

The remaining three combinations claim existence of fixed points of a progressive map on a chain-complete poset, a progressive map on a directed-complete poset, and a monotone map on a chain-complete poset. The first of these is the Bourbaki-Witt principle, which we study in this paper. Judging from Theorem 3.2, one might suspect that the second would have an intuitionistic proof, but in fact Dacar [4] has observed that it is equivalent to the Bourbaki-Witt principle.

Theorem 3.3 (Dacar) *The following are intuitionistically equivalent:*

1. *Any progressive map on a chain-complete poset has a fixed point above every post-fixed point.*

2. *Any progressive map on a directed-complete poset has a fixed point above every post-fixed point.*

Proof. The direction from chain-complete posets to directed-complete ones is straightforward: if P is directed-complete and x is post-fixed for a progressive $f : P \to P$, then $\{y \in P \mid x \leq y\}$ is chain-complete and closed under f.

To prove the converse, suppose the statement holds for directed-complete posets, and let $f : P \to P$ be a progressive map on a chain-complete poset P. The set C of chains in P, ordered by inclusion, is directed-complete. The map $F : C \to C$, defined by $F(A) = A \cup f(\sup A)$, is progressive, so has a fixed point B above $\{x\}$. Now $f(\sup B) \in B$ and hence $f(\sup B) \leq \sup B$, showing that $\sup B$ is a fixed point of f above x. □

The last combination is the Knaster-Tarski principle for chain-complete posets:

> *"A monotone map on a chain-complete poset has a fixed point above every post-fixed point."*

Most of what we show for the Bourbaki-Witt principle in this paper holds almost without alteration for the Knaster-Tarski principle, with one notable exception. As we saw in Theorem 3.2, the directed-complete version of the Knaster-Tarski principle is intuitionistically provable, while the directed-complete version of the Bourbaki-Witt theorem fails in general, as we will see in Section 4.

Finally, looking at the relationship *between* the Knaster-Tarski and Bourbaki-Witt principles, we have:

Proposition 3.4 *The Bourbaki-Witt principle implies the Knaster-Tarski principle.*

Proof. Let $f : P \to P$ be a monotone map on a chain-complete poset P, and suppose $x \leq f(x)$. Say that a chain $C \subseteq P$ is *nice* if f is progressive on C.

Then the poset of nice chains under inclusion is chain-complete (indeed, directed-complete) and has a progressive map s, which sends C to

$$s(C) = C \cup \sup\{f(y) \mid y \in C\}.$$

The Bourbaki-Witt principle gives a fixed-point C of s above $\{x\}$. Then $\sup C$ is a fixed point of f above x. $\square$

We do not know whether this implication can be reversed!

We summarize the intuitionistic provability of the six variants, and implications between them, in the following diagram (where ✓ stands for "provable"):

$$\begin{array}{lccccc}
 & \text{Complete} & \begin{array}{c}\text{Directed-}\\\text{complete}\end{array} & & \begin{array}{c}\text{Chain-}\\\text{complete}\end{array} \\
\text{Progressive} & \checkmark & \text{✗} & \Longleftrightarrow & \text{✗} \\
 & & & & \Downarrow \\
\text{Monotone} & \checkmark & \checkmark & & \text{✗}
\end{array}$$

3.2 Equivalent forms of Bourbaki-Witt

Bourbaki-Witt may be stated in several slightly different forms, all classically equivalent. In fact, they turn out to be intuitionistically equivalent as well.

Theorem 3.5 *The following are intuitionistically equivalent:*

1. *Any progressive map on a chain-complete poset has a fixed point above every point.*
2. *Any progressive map on a chain-complete poset has a fixed point.*
3. *Every chain-complete poset has a fixed-point operator for progressive maps.*

Proof. Let us first establish the equivalence of the first two statements. Every chain-complete poset has a least element, the supremum of the empty chain, above which one may seek fixed points. Conversely, a fixed-point of a progressive map $f : P \to P$ above $x \in P$ the same thing as a fixed-point of f restricted to the chain-complete subposet $\uparrow x = \{y \in P \mid x \leq y\}$.

The third statement clearly implies the second one. Conversely, suppose the second statement holds. Take any chain-complete poset P and let

$\mathrm{Prog}(P)$ be the set of progressive maps on P. We can endow the exponential $P^{\mathrm{Prog}(P)}$ with a chain-complete partial order, defined by

$$\langle x_f \rangle \leq \langle y_f \rangle \iff \forall\, f \in \mathrm{Prog}(P) \,.\, x_f \leq y_f,$$

where we write $\langle x_f \rangle$ for the element of $P^{\mathrm{Prog}(P)}$ that maps f to x_f. The endomap $h : P^{\mathrm{Prog}(P)} \to P^{\mathrm{Prog}(P)}$,

$$h(\langle x_f \rangle) = \langle f(x_f) \rangle, \tag{1}$$

is progressive, and so has a fixed point, which is exactly the desired fixed-point operator. $\square$

Any of the the statements from Theorem 3.5 may be interpreted in the internal language of a topos $\mathcal{E}$. When we do so we refer to them as the *internal* Bourbaki-Witt principle. One may also consider *external* versions in which the universal quantifiers range externally over progressive morphisms, rather than internally over the object of progressive maps. A morphism $f : P \to P$ is *progressive* if it is so in the internal logic; equivalently, if $(\mathrm{id}_P, f) : P \to P \times P$ factors through $\leq$, viewed as a subobject of $P \times P$.

Theorem 3.6 *The* internal *and* external *Bourbaki-Witt theorems are equivalent in a topos $\mathcal{E}$:*

1. *Internal: for every chain-complete poset P in $\mathcal{E}$, the statement*

   $$\forall\, f \in P^P \,.\, (\forall\, x \in P \,.\, x \leq f(x)) \Rightarrow \exists\, x \in P \,.\, f(x) = x.$$

   is valid in the internal logic of $\mathcal{E}$.

2. *External: for every chain-complete poset P in $\mathcal{E}$ and every progressive morphism $f : P \to P$ the internal statement $\exists\, x \in P \,.\, f(x) = x$ is valid.*

Proof. The internal form obviously implies the external one. Conversely, suppose the external form holds, and consider any chain-complete poset P in $\mathcal{E}$. As in the proof of Theorem 3.5, we may construct in $\mathcal{E}$ the chain-complete poset $P^{\mathrm{Prog}(P)}$, and the canonical progressive morphism h thereon. By (2), the statement $\exists\, z \in P^{\mathrm{Prog}(P)} \,.\, h(z) = z$ holds in $\mathcal{E}$. We now conclude, just as in the proof of Theorem 3.5, that there exists in the internal sense a fixed-point operator for P, which implies the internal form. $\square$

Similarly, various forms of the *failure* of Bourbaki-Witt turn out to be equivalent. The failure of a universal statement is generally weaker, intuitionistically, than the existence of a specific counterexample; and for the negation of the full, unbounded Bourbaki-Witt principle, this seems to be the case. (Indeed, in topos logic, with no unbounded quantifiers, this negation cannot even be stated.) However, as soon as the failure is in any way *bounded*, one can construct a counterexample.

Theorem 3.7 *The following are intuitionistically equivalent:*

1. *There is a chain-complete poset and a progressive map on it which has no fixed points.*
2. *There is a chain-complete poset on which not every progressive map has a fixed point.*
3. *There is a set $\mathcal{W}$ of chain-complete posets such that not every progressive map on every poset in $\mathcal{W}$ has a fixed point.*

Proof. Clearly, the first statement implies the second one, which implies the third. To close the circle, suppose $\mathcal{W}$ is a set of chain-complete posets as in the third statement. Then the chain-complete poset $\prod_{P\in\mathcal{W}} P^{\mathrm{Prog}(P)}$ carries a progressive endomap with no fixed point, sending F to $(P, f) \mapsto f(FPf)$. □

We remark that the key ingredient in most proofs from this subsection was that any product of chain-complete partial orders is again chain-complete. Lemma 4.4 below may be seen as a strong generalisation of this fact.

3.3 A set of all trichotomous ordinals?

In the (futile) search for an intuitionistic proof of the Bourbaki-Witt theorem it seems natural to consider the transfinite iteration of a progressive map $f : P \to P$,

$$x \leq f(x) \leq f^2(x) \leq \cdots \leq f^{\omega}(x) \leq f^{\omega+1}(x) \leq \cdots$$

One feels that a fixed point will be reached, if only we can produce a sufficiently long order to iterate along. In classical set theory this is possible, even without the axiom of choice. For example, Lang [7] proves the Bourbaki-Witt theorem by considering the least subset $C \subseteq P$ which contains x, is closed under f and under suprema of chains. He proves, classically but without choice, that C is a chain, from which it quickly follows that the supremum $\bigvee C$ is a fixed point of f. In fact, C is (isomorphic to) an ordinal and is precisely large enough for the iteration of f to stabilise after C-many steps.

Can fixed points always be found by transfinite iteration, as long as they exist? Is failure of Bourbaki-Witt always due to a lack of existence of long enough ordinals? In Subsection 4.4 below, we answer the first question negatively: there is a topos in which the Bourbaki-Witt principle holds, but fixed points cannot generally be reached by iteration along ordinals. In this section, we show that the answer to the second question is positive: the Bourbaki-Witt principle fails precisely when there is a set of all ordinals.

In the intuitionistic world the matter is complicated by the fact that the intuitionistic theory of ordinals is not nearly so well behaved as the classical; see [13] for an analysis of what can be done. Thus, before proceeding, we need to pick a definition of ordinals.

Recall that a relation $<$ on L is *inductive* if it satisfies the induction principle

$$(\forall\, x \in L\, .\, (\forall\, y < x\, .\, \phi(y)) \Rightarrow \phi(x)) \implies \forall\, x \in L\, .\, \phi(x),$$

for all predicates ϕ on L. In addition to the induction principle for predicates, an inductive relation admits inductive definitions of maps. However, in our case, attempting to iterate a progressive map, there is a complication. Given a progressive map $f : P \to P$ on a chain-complete poset P, we would like to define $\tilde{f} : L \to P$ inductively by

$$\tilde{f}(y) = \bigvee\nolimits_{x<y} f(\tilde{f}(x)).$$

For this to be a valid definition we need to know that these suprema exist, so we must ensure inductively that each $\{f(\tilde{f}(x)) \mid x < y\}$ is a chain in P. A fairly strong notion of ordinals is needed:

Definition 3.8 A *trichotomous ordinal* $(L, <)$, is a transitive inductive relation satisfying the law of trichotomy: for all $x, y \in L$, either $x < y$, $x = y$, or $y < x$.

One can now show:

Lemma 3.9 *If L is a trichotomous ordinal, and f is a progressive map on a chain-complete poset P, then we may define the iteration $\tilde{f} : L \to P$ of f along L as described above, by the equation $\tilde{f}(y) = \bigvee_{x<y} f(\tilde{f}(x))$.*

Proof. By induction on y, $\tilde{f}$ is monotone whenever it is defined; so $\{\tilde{f}(x) \mid x < y\}$ is always a chain in P, and thus $\tilde{f}$ is totally defined on L. □

A few more observations about trichotomous ordinals, similarly straightforward by induction, will also be useful:

1. A inductive relation is asymmetric—that is, $(x < y) \Rightarrow \neg(y < x)$ for all x, y—and irreflexive.

2. Trichotomous ordinals are rigid: the only automorphism $L \to L$ is the identity.

3. The class of trichotomous ordinals forms a pre-order under the "embeds as an initial segment" relation, and is moreover chain-complete.

4. If L is a trichotomous ordinal, then so is the strict order $L + 1$ formed by adjoining a new top element above L. This ordinal is called the *successor* of L; the successor map on the class of trichotomous ordinals is progressive and has no fixed point. Note that unlike classically, the successor map may *not* be monotone [13].

We are now equipped to compare ordinal existence and Bourbaki-Witt as promised.

Theorem 3.10 *The following are (intuitionistically) equivalent:*

1. *There is a progressive map on a chain-complete poset which has no fixed point.*

2. *There is a set into which every trichotomous ordinal injects.*

3. *There is a set $\mathcal{O}'$ of trichotomous ordinals such that every trichotomous ordinal is isomorphic to some ordinal in $\mathcal{O}'$.*

4. *There is a set $\mathcal{O}$ of trichotomous ordinals such that every trichotomous ordinal is isomorphic to a unique ordinal in $\mathcal{O}$. (In topos-theoretic terms, $\mathcal{O}$ is a classifying object for trichotomous ordinals.)*

Proof. We prove four implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

First, suppose P is chain-complete and $f : P \to P$ is a progressive map without fixed points. For any trichotomous ordinal L, we can define the iteration $\tilde{f}$ of f along L as described above. But now, the map $\tilde{f}$ is injective: if $\tilde{f}(x) = \tilde{f}(y)$, then $x < y$ cannot hold because that would give us a fixed point of f:

$$f(\tilde{f}(x)) \leq \bigvee\nolimits_{x<y} f(\tilde{f}(x)) = \tilde{f}(y) = \tilde{f}(x) \leq f(\tilde{f}(x)).$$

The case $y < x$ is similarly impossible, so $x = y$. Thus every trichotomous ordinal embeds in P, as required.

Second, if every ordinal injects into a set A, then take

$$\mathcal{O}' = \{(L, <) \in \mathcal{P}(A) \times \mathcal{P}(A \times A) \mid (L, <) \text{ is a trichotomous ordinal}\}.$$

In the third implication we avoid the axiom of choice by using an idea familiar from the construction of moduli spaces in geometry: if we can weakly classify a class of objects and they have no non-trivial automorphisms, then we can classify them. Take the quotient set $\mathcal{O}'/{\cong}$ of equivalence classes of ordinals up to isomorphism. Now for any equivalence class $C \in \mathcal{O}'/{\cong}$, we can define a canonical representative as follows. Take the coproduct $S_C = \coprod_{L \in C} L$, and for $L, L' \in C$, $x \in L$, $y \in L'$, set $x \sim y$ if the unique isomorphism $L \cong L'$ sends x to y. Then $R_C = S_C/{\sim}$ has a natural bijection to each $L \in C$, commuting with the isomorphisms between these; so with the ordering transferred along any of these bijections, R_C is a trichotomous ordinal, and a representative for C. Thus $\mathcal{O} = \{R_C \mid C \in \mathcal{O}'/{\cong}\}$ is as desired.

The last implication is easy because the set $\mathcal{O}$ of trichotomous ordinals, if it exists, is a chain-complete poset under the initial-segment preorder; and the successor map on $\mathcal{O}$ is progressive and has no fixed points. $\square$

4 Topos models

4.1 Bourbaki-Witt fails in realisability toposes

The Bourbaki-Witt principle fails in the effective topos Eff, as was shown by the first author [2]. We indicate how the proof can be adapted easily to work in any realisability topos. For background on realisability see [14].

Let A be a partial combinatory algebra and $\mathsf{RT}(A)$ the realisability topos over it. The category of sets Set is equivalent to the category of sheaves in $\mathsf{RT}(A)$ for the $\neg\neg$-coverage. The inverse image part of the inclusion $\mathsf{RT}(A) \to \mathsf{Set}$ is the global points functor $\Gamma : \mathsf{RT}(A) \to \mathsf{Set}$, and we denote the direct image by $\nabla : \mathsf{Set} \to \mathsf{RT}(A)$.

Let κ be the cardinality of A, where we work classically in Set. The successor κ^+ is a regular cardinal, which we view as an ordinal. The successor map $s : \kappa^+ \to \kappa^+$ is progressive and monotone but has no fixed points. This is no suprise as κ^+ is not chain-complete, although it has suprema of chains whose cardinality does not exceed κ. But the poset $\nabla\kappa^+$ *is* chain-complete in $\mathsf{RT}(A)$ because every chain in $\mathsf{RT}(A)$ has at most κ elements (to see what exactly this means in the internal language of $\mathsf{RT}(A)$ consult [2]), therefore the successor map $\nabla s : \nabla\kappa^+ \to \nabla\kappa^+$ provides a counterexample to both the Bourbaki-Witt and the Knaster-Tarski principle.

In the effective topos Eff the object $\mathcal{O}'$ from Theorem 3.10 has a familiar description. It is none other than Kleene's universal system of notations O for recursive ordinals, see [10, 11.7].

4.2 Bourbaki-Witt does not hold in the free topos

Recall [6] that there is an elementary topos $\mathcal{E}_{\mathit{free}}$, "the free topos", constructed from the syntax of intuitionistic higher-order logic (IHOL), and pseudo-initial in the category of elementary toposes and logical morphisms. Objects in $\mathcal{E}_{\mathit{free}}$ are thus exactly such objects as are definable in IHOL, and have exactly such properties as are provable.

Does Bourbaki-Witt hold in the free topos? It cannot *fail*, since the canonical logical morphism $\mathcal{E}_{\mathit{free}} \to \mathsf{Set}$ would preserve any failure. But it might not hold either: there could be some poset defined in IHOL, provably chain-complete, with a definable and provably progressive map, for which the existence of a fixed point is not provable. To show this unprovability for some particular P and f, it suffices to give a topos $\mathcal{E}$ in which the interpretation of f has no fixed point. Happily, with just a little work, the poset $\nabla\omega_1$ in Eff (an instance of the construction of Subsection 4.1), and its successor map, can be exhibited as such an interpretation.

Theorem 4.1 *The Bourbaki-Witt principle does not hold in $\mathcal{E}_{\mathit{free}}$.*

Proof. As we saw above, ∇ embeds Set as sheaves for the $\neg\neg$ topology on Eff. In Set, ω_1 is definable as a subquotient of $2^{\mathbb{N}}$: the set of all subsets of $\mathbb{N} \times \mathbb{N}$ describing well-orderings of $\mathbb{N}$, modulo isomorphism of the resulting well-orders. Thus, interpreting this definition in the Kripke-Joyal semantics for $\neg\neg$ in Eff, $\nabla(\omega_1)$ is definable as the $\neg\neg$-sheafification of a certain quotient of a certain subobject of ${\Omega_{\neg\neg}}^{\mathbb{N}\times\mathbb{N}}$; similarly, its order and the successor map are definable, so we have a poset $\omega_1^{\neg\neg}$ in $\mathcal{E}_{\mathit{free}}$, together with a progressive endomap s, which are interpreted as $\nabla(\omega_1)$ and its successor map in Eff.

Unfortunately, $\omega_1^{\neg\neg}$ cannot be chain-complete in $\mathcal{E}_{\mathit{free}}$, since in Set it is interpreted as ω_1. We can remedy this, however, using an exponenential by a truth-value. Let t denote the set $\{* \in 1 \mid \omega_1^{\neg\neg}$ is chain-complete$\}$, and set

$$P := (\omega_1^{\neg\neg})^t = \prod_{u\in t} \omega_1^{\neg\neg}.$$

This now has a natural chain-complete ordering, since the second description exhibits it as a dependent product of chain-complete posets: $\omega_1^{\neg\neg}$ is not in general chain-complete, but given any $u \in t$, it certainly is! Similarly, the endomap $f = s^t$ is progressive. But in Eff, the truth-value in question is 1, so P is interpreted as $\nabla(\omega_1)^1 \cong \nabla(\omega_1)$, and f as successor. Thus the existence of a fixed point of f is not provable, so we have a non-example of Bourbaki-Witt in $\mathcal{E}_{\mathit{free}}$. $\square$

Taking exponentials by truth-values in this fashion may be seen as an intuitionistic implementation of the classical construction "if P is chain-complete then P, else 1".

4.3 Bourbaki-Witt holds in cocomplete toposes

We have seen that the Bourbaki-Witt and Tarski conditions are not in general constructively valid. However, they hold in an important class of models thanks to the following transfer principle.

Theorem 4.2 *If $\mathcal{E} \to \mathcal{F}$ is a geometric morphism and $\mathcal{F}$ satisfies the Bourbaki-Witt principle, then so does $\mathcal{E}$.*

In particular, any cocomplete topos $\mathcal{E}$ has a geometric morphism (Γ, Δ) : $\mathcal{E} \to \mathsf{Set}$, where $\Gamma(A) = \mathcal{E}(1, A)$ is the global-points functor and $\Delta(X) = \coprod_X 1$ takes a set X to the X-fold coproduct of 1's. By applying the theorem to this case, we see that the Bourbaki-Witt principle holds in cocomplete toposes:

Corollary 4.3 *Any cocomplete topos, in particular any sheaf topos, satisfies the Bourbaki-Witt principle.*

Since this is our guiding example, we will write the geometric morphism as (Γ, Δ) in general, for the comforting familiarity it provides. To prove the theorem one requires a main lemma:

Lemma 4.4 *If $(\Delta, \Gamma) : \mathcal{E} \to \mathcal{F}$ is a geometric morphism and P is chain-complete in $\mathcal{E}$, then $\Gamma(P)$ is chain-complete in $\mathcal{F}$.*

Proof. We wish to construct a supremum map $\bigwedge_{\Gamma P}$: $\mathrm{Ch}(\Gamma P) \to \Gamma P$. Consider the universal chain in ΓP, i.e. the $\mathrm{Ch}(\Gamma P)$-indexed subset of ΓP

$$C = \{(x, c) \mid c \in x\} \hookrightarrow \mathrm{Ch}(\Gamma P) \times \Gamma P.$$

ΔC is now a $\Delta(\mathrm{Ch}(\Gamma P))$-indexed subset of $\Delta\Gamma P$, and indeed is a chain, since Δ preserves $\vee$; so its image $\widehat{C}$ under $\epsilon_P : \Delta\Gamma P \to P$ (the co-unit of the geometric morphism) is a $\Delta(\mathrm{Ch}(\Gamma P))$-indexed chain in P. Thus there is a map $s : \Delta(\mathrm{Ch}(\Gamma P)) \to P$ giving suprema for $\widehat{C}$, and hence for $\Delta(C)$.

Its transpose $\check{s} : \mathrm{Ch}(\Gamma P) \to \Gamma P$ is our candidate for $\bigwedge_{\Gamma P}$. We just need to show that $\mathcal{E}$ validates "for all $c : \mathrm{Ch}(\Gamma P)$ and $x : P$, $\check{s}(c) \leq x \Leftrightarrow \forall\, y \in c \,.\, (y \leq x)$", or in other words, that for any $(c, x) : A \to \mathrm{Ch}(\Gamma P) \times \Gamma P$, the map

$$(\check{s} \circ c, x) : A \to \Gamma P \times \Gamma P$$

factors through $\Gamma(\leq)$ if and only if the map

$$m : C \times_{\mathrm{Ch}(\Gamma P)} A = \{(y, a) \mid y \in c(a)\} \to \Gamma P \times \Gamma P$$

sending (y, a) to $(y, x(a))$ factors through $\Gamma(\leq)$.

But by the universal property of the adjunction, $(\check{s} \circ c, x)$ factors through $\Gamma(\leq)$ if and only if its transpose

$$\widehat{(\check{s} \circ c, x)} = ((s \circ \Delta(c)), \hat{x}) : \Delta(A) \to P \times P$$

factors through $\leq$. Since s gives suprema for $\Delta(C)$, this in turn happens if and only if the map

$$\hat{m} : \Delta(C) \times_{\Delta\mathrm{Ch}(\Gamma P)} \Delta(A) \to P \times P$$

sending (y, a) to $(y, \hat{x}(a))$ factors through $\leq$. But $\hat{m}$ is just the transpose of m, and so $\hat{m}$ factors through $\leq$ exactly if m factors through $\Gamma(\leq)$. Thus $\check{s}$ gives suprema for chains in ΓP, as desired. $\square$

Proof of Theorem 4.2. Suppose now that P is a chain-complete poset in $\mathcal{E}$, $f : P \to P$ is progressive, and $\mathcal{F}$ satisfies the Bourbaki-Witt principle.

$\Gamma(P)$ is chain-complete, by Lemma 4.4, and $\Gamma(f)$ is progressive, so $\mathcal{F}$ validates "$\Gamma(f)$ has some fixed point in $\Gamma(P)$". Being a statement of geometric logic, this is preserved by Δ, so $\mathcal{E}$ validates "$\Delta(\Gamma(f))$ has some fixed point in $\Delta(\Gamma(P))$".

But now $\epsilon_P \circ \Delta(\Gamma(f)) = f \circ \epsilon_P$ (by naturality), so if $x \in \Delta(\Gamma(P))$ is any fixed point of $\Delta(\Gamma(f))$, then $\epsilon_P(x) \in P$ is a fixed point of f. So $\mathcal{E}$ validates "f has some fixed point in P", as desired. $\square$

The only point in this section at which classical logic is required is for Corollary 4.3, to know that the Bourbaki-Witt theorem holds in Set.

4.4 Bourbaki-Witt does not imply ordinal existence

In Subsection 3.3 above, we asked: if Bourbaki-Witt holds, can any fixed point be computed by some long enough ordinal iteration? Here, we present a counterexample: a topos in which Bourbaki-Witt holds, but there are not enough ordinals to compute fixed points.

The rough idea is as follows: we first consider ordinals and posets in the presheaf topos $\mathsf{Set}^{\cdot\to\cdot}$, where an ordinal turns out to be a pair of ordinary ordinals with a strictly monotone map between them, written as $[L_1 \to L_0]$. Since in any ordinal, $<$ implies $\neq$, the length of the first component L_1 is bounded by the length of its second component L_0. By contrast, looking at chain-complete posets $[P_1 \to P_0]$ with progressive maps, the length of iteration required to find fixed points can be made arbitrarily large by blowing up just P_1, while holding P_0 fixed.

So in any assignment $(P, f) \mapsto L$ providing ordinals to compute fixed points, L_0 must depend on P_1, not only on P_0. But in any purely logical (i.e. IHOL) construction, L_0 would depend only on P_0, by the construction of the logical structure in $\mathsf{Set}^{\cdot\to\cdot}$. So although $\mathsf{Set}^{\cdot\to\cdot}$ has enough ordinals to compute fixed points, this fact cannot be realised by any purely logical construction.

Thus in $\mathcal{E}_{BW}[P, f]$, the free topos satisfying the Bourbaki-Witt principle and with a distinguished chain-complete poset and monotone map, there cannot be any ordinal computing the fixed point of f, since this would give a logical construction of such ordinals in any other topos, which we have seen is not possible.

We now formalise this argument, first setting up some terminology for the eventual goal.

Definition 4.5 Say that a topos $\mathcal{E}$ satisfying the Bourbaki-Witt principle *has enough ordinals* if for any chain-complete poset P in $\mathcal{E}$ with a progressive map f, there is some object B, inhabited in the internal sense (i.e. $B \to 1$ is epi), and some B-indexed family of ordinals $\langle L_b \mid b \in B \rangle$, such that $\mathcal{E}$ validates "for each $b \in B$, the iteration f^{L_b} of f along L_b has as its supremum a fixed point of f". (We say that the ordinals L_b *compute fixed points* for f.)

Definition 4.6 Let $\mathcal{L}_{BW}[P, f]$ be the theory in IHOL given by adding to pure type theory an axiom schema asserting that the Bourbaki-Witt principle holds, together with a new type P_{univ}, constants $\leq$ and f_{univ}, and axioms asserting that $(P_{univ}, \leq)$ is a chain-complete poset and f_{univ} a progressive map thereon. Let $\mathcal{E}_{BW}[P, f]$ be the syntactic topos of this type theory [6, II.11–16].

The universal property of $\mathcal{E}_{BW}[P, f]$ tells us that given any topos $\mathcal{F}$ satisfying Bourbaki-Witt and a progressive endomap f on a chain-complete poset P therein, there is a logical functor $\mathcal{E}_{BW}[P, f] \to \mathcal{F}$, unique up to canonical natural isomorphism, sending P_{univ} and f_{univ} to P and f respectively.[1]

The goal of this section is now:

Theorem 4.7 *The topos $\mathcal{E}_{BW}[P, f]$ does not have enough ordinals. In particular, there is no inhabited family of ordinals $L \to B \twoheadrightarrow 1$ that computes fixed points for f_{univ}.*

As indicated above, we begin by investigating ordinals and partial orders in $\mathsf{Set}^{\cdot\to\cdot}$. We will write objects of $\mathsf{Set}^{\cdot\to\cdot}$ as $X = [X_1 \to X_0]$. Since the functors $\mathrm{ev}_1, \mathrm{ev}_0 \colon \mathsf{Set}^{\cdot\to\cdot} \to \mathsf{Set}$ preserve finite limits, we may similarly write any (stric) partial order P in $\mathsf{Set}^{\cdot\to\cdot}$ as a map $[P_1 \to P0]$ of (strict) partial orders in Set.

The functor ev_0 is moreover logical; so if L is an ordinal in $\mathsf{Set}^{\cdot\to\cdot}$, then L_0 is an ordinal in Set. More generally, any slice functor $\mathrm{ev}_0 /X \colon \mathsf{Set}^{\cdot\to\cdot}/X \to \mathsf{Set}/X_0$ is logical; so if $L \to B$ is a family of ordinals in $\mathsf{Set}^{\cdot\to\cdot}$, then $L_0 \to B_0$ is a family of ordinals $\langle (L_0)_b \mid b \in B_0 \rangle$.

Lemma 4.8 *For any ordinal α in Set, let P_α be the poset $[\alpha + 1 \to 1]$ in $\mathsf{Set}^{\cdot\to\cdot}$, and f_α the progressive endomap of P_α that acts as successor on α, and as the identity elsewhere. Then:*

1. *P_α is chain-complete in $\mathsf{Set}^{\cdot\to\cdot}$; and*

2. *if $L \to B$ is any inhabited family of ordinals computing fixed points for f_α, then $\sup_{b \in B_0} (L_0)_b > \alpha$. In other words, with P_α, we have succeeded in blowing up the required length of L_0 to $\alpha+1$, while holding P_0 constant at 1.*

Proof. Chain-completeness follows immediately from Lemma 4.4, since $\alpha+1$ is chain-complete in Set, and the functor sending a set X to $[X \to 1]$ is the forward image of a geometric morphism, with inverse image ev_1.

[1]Contrary to what one might at first expect, these will not be the only logical functors out of $\mathcal{E}_{BW}[P, f]$; the axiom schema only forces Bourbaki-Witt to hold for posets in the image of the functor, not in the whole target topos.

Explicitly, the object of chains in P_α is given by

$$\mathrm{Ch}(P_\alpha) = \mathcal{P}(P_\alpha) \cong [\{S, T \mid S \subseteq \alpha+1,\, T \subseteq 1,\, \mathrm{im}(S) \subseteq T\} \to \mathcal{P}(1)$$

and the supremum map $\sup\colon \mathrm{Ch}(P_\alpha) \to P_\alpha$ is given by

$$\sup_1(S, T) = \sup S \in \alpha + 1 \qquad \sup_0(T) = * \in 1.$$

With this in hand, suppose $L \to B$ is some inhabited family of ordinals computing fixed points for f_α, and let $\tilde{f}\colon L \to P_\alpha$ denote the iteration of f_α along L.

For any ordinals α, β in Set, there is a canonical map of partial orders $\alpha \to \beta + \{\top\}$, the initial-segment embedding if $\alpha \leq \beta$, or stabilising at the top if $\alpha > \beta$. (One may regard this as a truncated rank function.) Viewing L_0 as the disjoint union of the ordinals $\langle (L_0)_b \mid b \in B_0 \rangle$, we obtain a notion of $(\alpha + \{\top\})$-valued rank for elements of L_0, and hence of L_1:

$$\mathrm{rk}\colon L_1 \to L_0 \to \alpha + \{\top\}.$$

This is very nearly strictly monotone: if $x < y$, then either $\mathrm{rk}(x) < \mathrm{rk}(y)$ or $\mathrm{rk}(x) = \mathrm{rk}(y) = \top$.

Now, we see that for every $x \in L_1$, $\tilde{f}_1(x) \leq \mathrm{rk}(x)$. If $\mathrm{rk}(x) = \top$, this is trivial; otherwise, we work by induction on $\mathrm{rk}(x)$:

$$\begin{aligned}
\tilde{f}_1(x) &= \sup_1(\{\tilde{f}_1(y) \mid y < x\}, \{\tilde{f}_0(z) \mid z < x|_0\}) \\
&= \sup\{\tilde{f}_1(y) \mid y < x\} \\
&\leq \sup\{\mathrm{rk}\, y \mid y < x\} \quad \text{(by induction)} \\
&\leq \mathrm{rk}\, x.
\end{aligned}$$

Now, by hypothesis, $\mathsf{Set}^{\cdot \to \cdot}$ validates "for each $b \in B$, $\sup\{\tilde{f}(i) \mid i \in L_b\}$ is a fixed point of f_α". Since B is inhabited, there is some $b \in B_1$; so calculating as above, we see that

$$\begin{aligned}
\top &= \sup\{\tilde{f}_1(x) \mid x \in (L_1)_b\} \\
&\leq \sup\{\mathrm{rk}\, x \mid x \in (L_1)_b\} \\
&\leq \sup\{\mathrm{rk}\, i \mid i \in (L_0)_{b|_0}\},
\end{aligned}$$

whence the ordinal $(L_0)_{b|_0}$ must be at least α, as desired. □

Finally, let us prove Theorem 4.7. Let $L \to B$ be any inhabited family of ordinals in $\mathcal{E}_{BW}[P, f]$; we wish to show that L does not compute fixed points for f.

Coonsider the logical morphism $F_1 \colon \mathcal{E}_{BW}[P, f] \to \mathsf{Set}$ sending (P, f) to the terminal poset 1 and its unique endomap. This sends L to some inhabited family of ordinals $\langle \lambda_b \mid b \in F_1(B) \rangle$; let α be an ordinal greater than the supremum of this family.

Now consider the logical morphism $F_{P_\alpha} \colon \mathcal{E}_{BW}[P, f] \to \mathsf{Set}^{\cdot \to \cdot}$, sending (P, f) to (P_α, f_α). Since ev_0 is a logical morphism and $(P_\alpha)_0 = 1$, the universal property of $\mathcal{E}_{BW}[P, f]$ enforces that $\mathrm{ev}_0 \circ F_{P_\alpha} = F_1$. So, in particular, $F_{P_\alpha}(L)_0$ is again the family of ordinals $\langle \lambda_b \mid b \in F_1(B) \rangle$, with supremum less than α.

Thus, by Lemma 4.8, the family of ordinals $F_{P_\alpha}(L)$ cannot compute fixed-points for f_α in $\mathsf{Set}^{\cdot \to \cdot}$. So, since F_{P_α} is logical, L cannot compute fixed points for f in $\mathcal{E}_{BW}[P, f]$. But L was arbitrary; so no inhabited family of ordinals can suffice, and $\mathcal{E}_{BW}[P, f]$ does not have enough ordinals.

A word of caution is necessary here, however: all these negations have been *external*, so for all we know it could still be the case that $\mathcal{E}_{BW}[P, f]$ validates the double-negated version "L does not fail to compute fixed points for f", for some $L \to B$.

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